Angles and Similar Triangles

Pleasanton Math Circle

1 Angle Chasing

In this section we illustrate how we can use angles to prove results. The diagram below illustrates the fundamental results of angles.

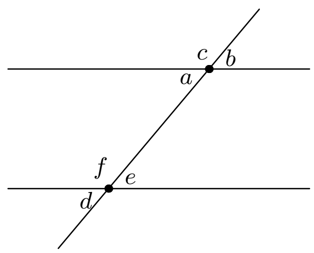

Figure 1: Angle Properties

Proposition 1.1 (Angle Properties)**.** In Figure 1, two of the lines are parallel. We know that $a = 180 - c$ and $a = d$. From this we can show that $a = b = d = e = 180 - c = 180 - f$.

Example 1.2. In $\triangle ABC$, $\angle A + \angle B + \angle C = 180°$.

Proof. Draw a line through A parallel to BC. Now use Proposition 1.1. □

Example 1.3. Let AB be a diameter of a circle and choose point C on this circle. Then $\angle C = 90°$.

1.1 Exercises

Exercise 1.4. One angle of a right triangle has measure x. Find the measures of the remaining two angles.

Exercise 1.5. One angle in an iscosceles triangle has measure $x°$. Find all possible measures of the other two angles.

Exercise 1.6. Show that the interior angles in a quadrilateral sum to 180°. Find a formula for an $n-$sided polygon.

Exercise 1.7. Points A, B, C are chosen on a circle with center O such that $ABCO$ is a rhombus. Find $\angle ABC$.

Exercise 1.8. In $\triangle ABC$, let the altitudes intersect at a point H (the orthocenter). Prove that $\angle BHC = 180 - \angle A$.

Exercise 1.9 (Inscribed Angle Theorem)**.** If A, B, C lie on a circle, then prove that $\angle ACB$ subtends an arc of measure $2\angle ACB$. Hint: Draw CO. (See Figure 2).

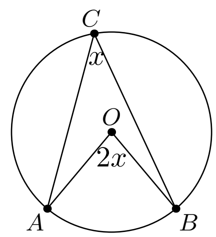

Figure 2: The Inscribed Angle Theorem

2 Similar Triangles

Identifying similar triangles are often key steps of a proof. Before discussing similar triangles, we consider an example using an even more primitive result - congruent triangles.

Proposition 2.1 (Congruency)**.** Two triangles are **congruent** if and only if they satisfy SSS, SAS, ASA, or HL.

Example 2.2. Given $\triangle ABC$, construct equilateral triangles $\triangle BCD, \triangle CAE, \triangle ABF$ outside of $\triangle ABC$. Prove that $AD = BE = CF$.

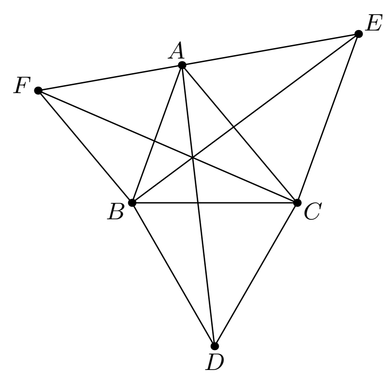

Figure 3: $AD = BE = CF$

Proof. Notice that $DC = BC$ and $CA = CE$. Also, $\angle DCA = 60^\circ + \angle C = \angle BCE$. So by SAS congruency, $\triangle DCA \equiv \triangle BCE$. Hence, $AD = BE$. Similarly, we can find $\triangle EAB \equiv \triangle CAF$, which implies $BE = CF$. Hence $AD = BE = CF$ as desired. □

Figure 3 suggests that AD, BE, CF are concurrent. In fact, they meet at the **first Fermat point**, which we will prove next meeting.

Proposition 2.3 (Similar Triangles)**.** Two triangles are **similar** if and only if they satisfy AA, SSS, SAS, ASA, or HL.

Proposition 2.4. In $\triangle ABC$, choose points X and Y on sides AB and AC. Then $\frac{AX}{AY} = \frac{AB}{BC}$ if and only if $XY \parallel BC$.

Proof. Verify that $\triangle AXY \sim \triangle ABC$ from SAS similarity. Then the result follows from the observation $\angle AXY = \angle ABC$. □

A noteworthy corollary is that if M and N are the midpoints of AB and AC, then $MN \parallel AC$. A nice application is the following.

Example 2.5 (Varignon Parallelogram)**.** Given a quadrilateral $ABCD$, let M, N, P, Q denote the midpoints of sides AB, BC, CD, DA respectively. Then $MNPQ$ is a parallelogram.

Proof. Draw in the diagonals and apply Proposition 2.4. □

Proposition 2.6. If the ratio of the perimeters of two similar figures is x, then the ratio of the areas of the two similar figures is x^2.

2.1 Exercises

Exercise 2.7. $\triangle ABC \sim \triangle DEF$. If $AB = 3, BC = 4, DE = 6$, find EF.

Exercise 2.8. In $\triangle ABC$, let M and N be the midpoints of AC and AB. Let BM and CN meet at G (the centroid). Prove that $BG : GM = 2 : 1$.

Exercise 2.9. Show that $ABCD$ is a parallelogram if and only if AC and BD bisect each other.

Exercise 2.10 (2015 Geometry Bee #6)**.** $\triangle ABC$ has area 100. Points P, Q, and R are on sides AB, BC, and CA respectively such that PQ is parallel to AC and RQ is parallel to AB. If $\triangle PQB$ has area 64, then find the area of $APQR$.

Exercise 2.11 (Angle Bisector Theorem)**.** In $\triangle ABC$, choose point D on segment BC such that $\angle BAD = \angle CAD$. Show that

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Hint: Use Figure 4

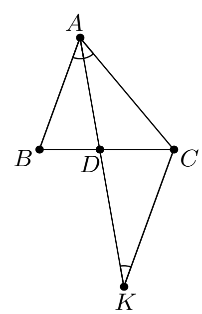

Figure 4: Angle Bisector Theorem

Exercise 2.12. In $\triangle ABC$ let $\angle C$ be right. Let D denote the foot of the altitude from C to AB. Then show that $\triangle ABC \sim \triangle ABC \sim \triangle CBD$. Use this to show that $AD \cdot BD = CD^2$.

Exercise 2.13. Prove that the area of a trapezoid with bases b_1 and b_2 and height h is $\frac{1}{2}(b_1 + b_2)h$.

Exercise 2.14. $ABCD$ is a trapezoid with $AB \parallel CD$ and area 32. Let AC and BD meet at X. $AB = 3$ and $CD = 5$. Find the area of $\triangle ABX$.

Exercise 2.15 (WOOT)**.** In the following diagram, E is the midpoint of CD, $EA = EB$, and $\angle FAC = \angle FBD = 90°$. Prove that triangles FAC and FBD are similar.