Random Card Shuffle

Pleasanton Math Circle

January 12st, 2017

1 The Top-in Shuffle

How do you gauge the randomness of a shuffle? In order to do his, we first need to understand what a **shuffle** is and what **randomness** means. Consider the function $f(x)$ which represents a permutation of a deck $ABCDE$.

An Example Permutation					
x	1	2	3	4	5
$f(x)$	2	3	1	5	4

In this shuffle, a card at position 1 will get moved to position 2. A card at position 2 will get moved to position 3, etc. Therefore, we will end up with $CABED$. A shuffle does not tell us anything about the cards in the deck itself. It only shows how the cards move. In shorthand, we write the permutation as $f(x) = [23154]$, or $(123)(45)$ using cyclic notation. The set of all permutations of a deck is defined to be S_n. A shuffle is simply a method that assigns probabilities to each permutation of a deck.
For example, consider a specific shuffle: the top-in shuffle. In this shuffle, you take the top card of the deck, then move it, with equal probability, to anywhere in the deck. This assignment of

The Top-in Shuffle					
permutation	[123]	[213]	[231]	[321]	[312]
probability	1/3	1/3	1/3	0	0

probability to the six possible permutations is called a probability density.

1. how many permutations does a deck with n cards have?
2. Create a shuffle chart for cutting a deck with 3 cards.
3. Think of a viable way to determine the randomness of a shuffle.

2 The Riffle Shuffle

The riffle shuffle is a model of a shuffle that someone in real life would do. In this shuffle, you split a deck into two smaller decks, with the probability of splitting a deck after k cards being $\binom{n}{k}/2^n$. Then, maintaining the relative order in each of the two individual decks, the cards are interlayed. For example, if a deck with 8 cards are split into two decks $1-4$ and $5-8$, some possibilities include 1627384 or 15678234, etc.

1. Prove that the probabilities of all the permutations sum to 1.
2. How many permutations does a deck with n cards with cut after k cards have?
3. Create a chart listing all possible permutations given a cut at position k.
4. What is the probability of achieving each cut+interweaving of the cards?
5. Create a shuffle chart for cutting a deck with 3 cards.

3 Determining the Randomness

Now, we are finally ready to gauge randomness. Consider two distinct shuffles with probability densities Q_1 and Q_2. The variation between the two shuffles is defined to be

$$||Q_1 - Q_2|| = 1/2 \sum_{x \in S_n} |Q_1(x) - Q_2(x)|.$$

Therefore, a good method to gauge randomness is the variation from a distribution where each permutation has equal probability, or

$$\sigma = 1/2 \sum_{x \in S_n} |Q(x) - 1/n!|$$

1. What is the range of the variation function?
2. What does a small σ indicate? A large σ?

4 How Many Shuffles?

Consider what we call a **Rising Sequence**. A rising sequence is a sequence in which we take a card in a deck, then find the consecutively next higher card, while proceeding right from the original card. For example, the sequence 45**1**6**23**78 has two rising sequences: 123 and 45678. Now, instead of the riffle shuffle, consider the a-shuffle, a modification of the riffle shuffle where we consider not splitting the deck into 2 smaller decks, bu a smaller decks, before interweaving the cards. Instead of $\binom{n}{k}/2^n$, the probability of splitting the deck into smaller decks of size $p_1, p_2, ..., p_n$ is $\frac{n!}{p_1!p_2!...p_a!}/a^n$.

Challenge Problem 1 Find that the probability of achieving each cut+interweaving of the cards

Challenge Problem 2 Prove that the probability of achieving a permutation with r rising sequences is $\binom{n+a-r}{n}/a^n$. Hint: start your casework on the riffle shuffle.

So, why did we generalize the riffle shuffle? Persi Draconis figured out the multiplication theorem for the a-shuffle.

Theorem 1 (Multiplication Theorem)
Carrying out a a-shuffle followed by a b-shuffle is the same as doing an ab- shuffle.

You can try testing this theorem experimentally. We will finally proceed to find the probability density of R_k, or doing k riffle shuffles. By repeatedly applying riffle shuffles, notice that, by using the multiplication theorem, this is equivalent of doing a $2 \cdot 2 \cdot 2 \cdot \ldots = 2^k$-shuffle.

Challenge Problem 3 Find σ for k riffle shuffle σ_{R_k} in terms of r, k, n and $A_{n,r}$, the permutation of n cards with r rising sequences (aka Eulerian numbers).

Although the equation derived looks very formidable, using computers, we see that σ starts rapidly decreasing at $k = 5$, and practically reaches 0 by $k = 11.k = 7$ seems to be a good middle-point for the cutoff. Additionally, analysis of the graph shows that when n is large, $k = \frac{3}{2} \log n$ suffices to get σ close to 0.