Counting and Probability

Pleasanton Math Circle

November 17th, 2016

1 Introduction

Counting and probability is a common subject in contest mathematics, but it is also a fundamental part of combinatorics, a field that is still being studied.

2 Counting

In this section, we will explore some fundamental counting problems.

1. How many 5 letter sequences can Kenny make if each letter must be A, B, C, D, or E and repeats are allowed?

2. April, Jonathan, Daniel, Steve, and Jeffery are racing to Chipotle. How many different ways can they finish?

3. 5 candidates are running for a committee position. If the committee has a president, a vice president, and a secretary, how many different committees can be chosen? Given 3 committee members, how many ways are there to choose a president, vice president, and secretary from the three? Of the original five candidates, how many ways are there to form a committee of 3 with no order?

4. Jeffrey has six Pokemon. However, he only wants three of them. How many ways can he choose the three Pokemon he keeps?

5. Mr. Lomas has 4 different geometry books, 3 different algebra books, and 2 different cooking books. How many ways can he order these books on a shelf, if books of the same subject must be next to each other.

6. Kenny interviews 25 students about their opinions on Jeffery and Jeffrey. 13 students like Jeffery. 14 students like Jeffrey. If everybody likes either Jeffery or Jeffrey, how many people like both Jeffery and Jeffrey?

7. How many ways can you rearrange the letters in $MISSISSIPPI$?

3 Probability

Probability isn't too different from counting. See if you can figure some of these problems out.

1. Kenny flips 3 coins. What is the probability that they all land heads?

2. Jonathan rolls two six sided dice. What is the probability that the product of the two numbers is odd?

3. 5 PMC goers out of 20 are going to be given starbursts. What is the probability that Jonathan gets one but Daniel does not?

4. Shawn flips 4 coins. What is the probability that he flips at least one head?

5. Samantha rolls 3 die. What is the probability that they all land on different numbers?

6. Amber and Parth each roll a six sided dice. What is the probability that Amber's number is larger than Parths?

4 Two Paradoxical Problems

1. (Monty Hall) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick, at random, Door 1, and the host, who knows what's behind the doors, opens Door 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Should you switch?

2. (Bert's Box) There are three boxes. One has two gold coins, one has two silver coins, and one has one of each. You randomly choose a box and take a coin out. Its gold! Find the probability that the other coin in the box is also gold.

5 More with Combinations

Officially, **combinations** are defined to be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This counts the number of ways to choose k objects from a set of n objects. Referring to our earlier questions, we can see that $\frac{n!}{(n-k)!}$ is like the number of ways to choose a committee with positions (president, vice president, etc). $k!$ is the number of ways to rearrange a k-person committee into an order, so it gets rid of the positions. Dividing them gives the number of ways to choose a committee with no order.

Each of the problems below can be solved (cleverly) with combinations. Can you figure out how?

1. Jeffery the Jangaroo is at the point $(0, 0)$. Each time he jumps, he either hops one unit right or one unit up. How many different ways can he reach the point $(5, 8)$? What if he must first take a rest stop at the point $(3, 4)$?

2. Tejasvi is giving away 6 pieces of candy to 4 students. How many ways can she distribute the candy? What if each student must get at least one piece of candy?

3. How many different ways can Mr. Lomas choose non-negative integers a, b, and c such that $a + b + c = 9$. What if a, b, and c are positive integers? What if they must all be odd positive integers? How many different ways can $a+b+c$ be strictly less than 9?

6 "States" and Infinite Probability

In section 3, you explored the probability of events that happen only a finite number of times. For example, Shawn flips 4 coins or Samantha rolls 3 dice. What happens if the events occur an infinite number of times? Consider this example:

Exercise 1. Daniel plays a game with Kenny. On the first turn, he wins with a 1/2 probability. Otherwise, it is Kenny's turn. on his turn, Kenny wins with 1/3 probability. If Kenny doesn't win, it is Daniel's turn again and the cycle repeats. What is the probability that Daniel wins?

Consider the perspective of Daniel. When it is his turn, he has a 1/2 probability of winning. Otherwise, there is a $(1/2)2/3 = 1/3$ chance of both Daniel and Kenny losing, making it Daniel's turn again. Thus, the probability of Daniel winning, p, is equivalent to $1/2 + 1/6p$. Solving gives $p = 3/5$. In this problem, notice that we had one *state*, Daniel's turn, and transitioned from this state to the next state. In fact, such transitions can be of one state to a completely different state as well.

Definition 2. A **state** is a position or a configuration in the problem Creating transitions between states creates a chain of linear equations to solve the problem.

Although the definition is not yet clear, it is easily clarified by an example. Unlike the previous example, we will attempt to solve a problem with multiple states.

Exercise 3. Freddy the Mouse is on the leftmost square of a row of three squares. On the left of Freddy is a cat and on the right of the row of squares is a slice of cheese. Freddy can move left and right, each with probability 1/2. What is the probability that he reaches the cheese and doesn't get eaten by the cat?

Define a state n to be the number of moves from the cheese. Freddy starts at state number 3, the cheese is at state 0 and the cat is at state 4. Let P_n to be the probability that he reaches the cheese from state n. There are only 2 possible ways to move between states: a move left or a move right. From state 3 Freddy can move to state 4 or state 2. Thus, $P_3 = 1/2P_4 + 1/2P_2$. Can you finish this exercise?

Below are more problems to be solved using the above concepts. Beware, some of them are challenging!

1. Jeffery the Phrog is on the middle square of a 5 square row. He can move left or right with probability 1/3 and 2/3 respectively. What is the probability that he jumps off the right end of the row of squares?
2. Kenny the Kat is playing Random Pong with Daniel the Dawg. The ball currently exists the middle point of a 6-square-high and infinite-square-long grid. When not on a wall, it moves up, down, left, or right, with probabilities 1/10, 2/10, 3/10, 4/10 respectively. What is the probability that the ball hits Kenny's goal, which is the bottom wall, before it hits Daniel's goal, which is the top wall?
3. April the Pineapple-Pen create a sequence of letters consisting of $P's$ and $A's$. What is the probability that she writes three $P's$ in a row before she writes 2 $A's$ in a row?
4. A happy-go-lucky man is 3 meters right of a 300 foot high cliff. Given that he moves left or right one meter with probability 1/5 and 4/5 respectively, what is the probability, given an infinite amount of time, that he falls offf the cliff and dies?