Game Theory

Pleasanton Math Circle
October 6th, 2017

1 More than just numbers

Today we will be focusing on learning how to play games (or rather to win games). In later lectures, we'll see that solving other math problems isn't too different than trying to win games. One of the best tips we can give you when solving problems is to not give up. Many problems can be solved by thinking for a sufficient amount of time.

2 Warmup

Play the following game with a partner. Rules:

1. Each player takes turns saying 1 or 2 numbers starting from the number 1.
2. The player that says the number 21 loses.

What can you learn from the game? Do any situations guarantee a win for a certain player? Can a player guarantee a win? Can you find a winning strategy for either player?

3 A Canada/USA Mathcamp Problem

The following problem appeared on the 2016 Canada/USA Qualifying Quiz. Try playing the game with a partner and discussing possible strategies.

Francisco and Savannah are playing a game with two tokens, which are placed on the squares of a rectangular grid of arbitrary size. The two tokens must be in different rows and columns. The players take turns moving a token of their choice to any different square (not necessarily just an adjacent one) satisfying the following constraints:

- Tokens can never be moved upward or to the right.
- The row ordering must be preserved: if one token is above the other, it must stay above the other.
- The column ordering must be preserved: if one token is to the right of the other, it must stay to the right of the other.

Francisco goes first. Whichever player has no legal moves (with either token) loses.

1. Suppose one token begins above and to the right of the other. (The constraints require that the two tokens stay in that order.) For which starting positions does Francisco win and for which does Savannah win? What is the winning strategy in each case?
2. Do the same analysis assuming one token begins below and to the right of the other. (The constraints require that the two tokens stay in that order.)

4 More Games

Try some of these other games.

Game 1. 20 points on the circumference of a circle are chosen. Each player selects 2 points and draws a line between them that does not intersect any of the existing lines. A player loses if he/she cannot move. Who wins?

Game 2. There are two piles of candy. One pile contains 20 pieces, and the other 21. Players take turns eating all the candy in one pile and separating the remaining candy into two (not necessarily equal) piles. (A pile may have 0 candies in it) The player who cannot eat a candy on his/her turn loses. Which player, if either, can guarantee victory in this game?

Game 3. (Dot's and Boxes) Find the winning strategy to a 4x4 Dot's and Boxes game.

Game 4. Starting from 2, the players take turns adding to the current number any positive whole number less than the current number. The player who reaches the number 1000 wins. Which player, if either, can guarantee victory in this game?

Game 5. A box has 300 matches. Players take turns removing no more than half the matches in the box. The player who can't move loses. Can one player guarantee a win, and if so, how? Generalize: what if there were n matches in the box?

5 USAMO 2016/6

Here is a challenge problem which appeared on the 2016 USA Math Olympiad. Don't be intimidated!

(USAMO 2016/6). Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again. We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?