Guarding a Military Base

Pleasanton Math Circle
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1 The Problem

You are the commander of a new military base (which is a polygon). You remember that you must hire guards to defend the base. To completely defend the base, the guards must be able to see every part of the building. Additionally, since the government has a shortage of money, you need to hire the minimum number of guards. The guards can stand anywhere inside the building and can see in all directions, but cannot move. What is the minimum number of guards that you must hire?

2 Exploration

Try to guard these military bases.

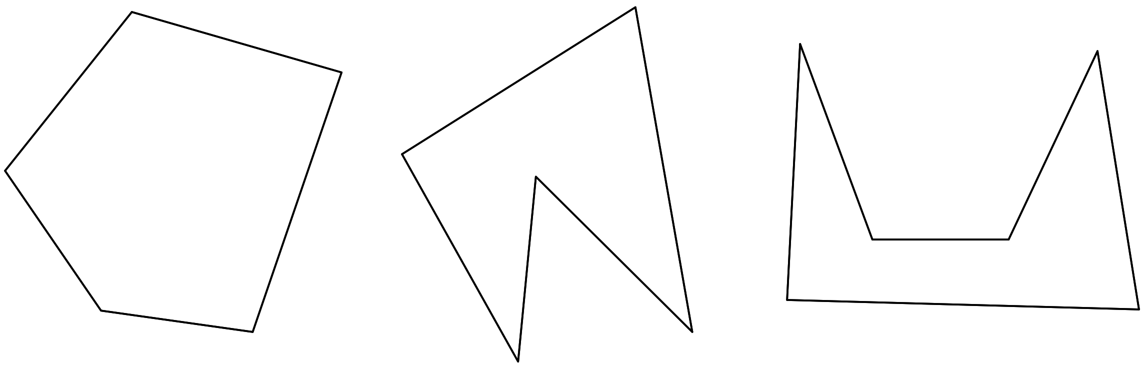

Figure 1: Military Bases

3 Helpful Questions

1. In a convex (all angles less than 180) military base, how many guards are necessary to supervise the entire base?
2. In the base furthest to the right, show that you have optimally guarded the military base. (Show that you can't do it with any fewer guards.)
3. Find a base that needs 3 guards. What is the fewest number of sides such a base has?
4. What would happen if the base has 5 sides? (Try drawing different pentagons.)
5. Is it better for the guard to be on the sides of the polygon, or inside the polygon?
6. Prove that you can always guard an n sided polygon with n guards. Can you improve this bound? (I.e. can you replace n with something lower, like $n-1$ or $\frac{n}{2}$?)

4 Theorem

This problem was first proposed in 1973 by Victor Klee and became known as the *Art Gallery Problem*. The answer to the question can be found in the following theorem. The first proof was given in 1975 by Vasek Chvatal, but the most famous (and beautiful) proof was produced in 1978 by Steve Fisk. In the following sections, you will reproduce his proof.

> **Theorem 1** (Art Gallery Theorem)
> Only $\lfloor \frac{n}{3} \rfloor$ guards or fewer are necessary to guard an $n-$sided polygon. Also, this is the best possible bound.

First lets make sure we understand the statement of the theorem. The floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 6.25 \rfloor = 6$.

Exercise 2. Compute $\lfloor 278.35 \rfloor, \lfloor 12 \rfloor$, and $\lfloor \pi \rfloor$.

The first part of the theorem states that given an $n-$sided polygon, $\lfloor \frac{n}{3} \rfloor$ guards is always enough (when placed carefully). For example, given an octagon, $\lfloor \frac{8}{3} \rfloor = 2$ guards is sufficient.

Exercise 3. Given that Theorem 1 holds, how many guards do we need for a 22 sided polygon?

In fact, we will prove this part of the theorem second. The second part of the theorem states that this is "the best possible bound." What this means is that you cannot replace $\lfloor \frac{n}{3} \rfloor$ in the theorem with anything lower. This is equivalent to the following proposition:

> **Proposition 4**
> There exists an $n-$sided polygon that *requires* $\lfloor \frac{n}{3} \rfloor$ guards.

Now we will show that this is true.

Exercise 5. Find a hexagon that requires 2 guards and a 9 sided polygon that requires 3 guards.

Exercise 6. Find a general construction of a $3n-$gon that requires n guards. (Hint: Try extending your constructions in Exercise 5.)

Exercise 7. Prove Proposition 4.

Now that we proved part two of the theorem, lets get back to the first part. First, we need a quick definition.

Definition 8. A **triangulation** of a polygon is made by using diagonals to divide the polygon into triangles like in the figure below.

In fact, any polygon can be triangulated. You can verify that this is intuitively correct. If you have time you can also find a rigorous proof. In our proof, we will use a lemma. A lemma is kind of like a mini theorem that we use to show that our main theorem is true.

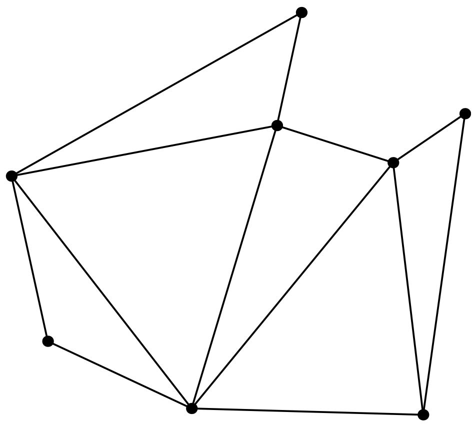

Figure 2: A triangulation

Lemma 9

Any triangulation of a polygon has a nice 3−coloring.

This just means that we can color the vertices of a polygon with three colors such that every triangle has all three colors. Equivalently, no vertices that are connected are colored the same. For an example, see the figure below.

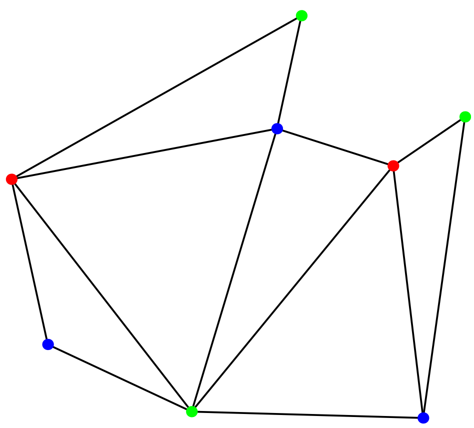

Figure 3: A nice 3−coloring

Exercise 10. Prove Lemma 9. First see that a triangle has a nice 3−coloring. Now using this, can you show that a quadrilateral has a nice 3−coloring. From this, show that you can show the result for a pentagon ... and just keep going.

We are actually almost done!

Exercise 11. Prove Theorem 1 using Lemma 9.

This problem has several variations and extensions. For example, what if the polygon has only right angles? Or what if the guards can move along a line segment. A good place to further explore these results is at `http://www.math.cornell.edu/~web401/mark.artgallery.pdf`