Proof Techniques — Solutions

Pleasanton Math Circle: Middle School

Ryan Vir

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§1 Warm-Up

Problem 1.1. See solution in section 2 of the handout.

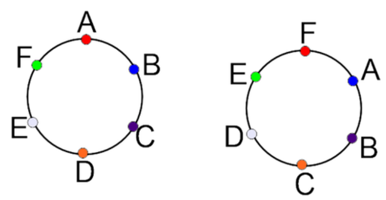
Problem 1.2. Assume for the sake of contradiction that $\sqrt{2}$ is rational. That means that it can be expressed as $\frac{a}{b}$, where a and b are coprime integers. This means that at least one of them is odd. So, $\frac{a}{b} = \sqrt{2} \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. This means that a is even, so $a = 2k$ for some integer k. So, $2b^2 = a^2 = (2k)^2 = 4k^2$, which simplifies to $b^2 = 2k^2$. This means that b must also be even, which contradicts that at least one of a and b are odd. Hence, we have our contradiction, so $\sqrt{2}$ must be irrational.

Problem 1.3. Assume for the sake of contradiction that there are a finite number of primes. Call them $p_1, p_2, \ldots, p_n$. Now, consider the number $P = p_1 p_2 \ldots p_n + 1$. P is not divisible by any of the primes p_i, so it must be divisible by some other prime. This contradicts our assumption that p_i are all the primes, so there must be an infinite amount of prime numbers.

Problem 1.4. See problem 4.3 for a proof of a generalized version of this.

§2 WLOG

Problem 2.2. Since it is a circular table, we have to account for rotations. For example, the following two seating arrangements would be considered identical on a six person table.



So, let's assume WLOG that in a group of persons A, B, C, D, E, F, and G, person A sits at one designated chair. Then, we only have to arrange the rest of the six people in the other six chairs. We can do this in $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = \boxed{720}$ ways.

§3 Contradiction

Problem 3.2. If x is 0, then $c = 0$, which is not allowed. If x is positive, then $ax^2 + bx + c > 0$, and cannot equal 0. Hence, any solution x must be negative.

Problem 3.3. A person can have $0, 1, 2, \ldots,$ or 7 friends in the group. If everyone had a different number of friends, then one person would have no friends and another would have 7 friends. But, this is a contradiction because one person having 7 friends implies that no one has no friends. So, there have to be at least two people with the same number of friends in the group.

§4 Induction

Problem 4.3. Keep in mind that here r is a constant, and we want to prove with induction on n. We begin by testing the identity for small values of n:

$$1 = \frac{r-1}{r-1}, 1+r = \frac{r^2-1}{r-1}, 1+r+r^2 = \frac{r^3-1}{r-1}.$$

We now assume the identity is true for an arbitrary $n = k$:

$$1 + r + r^2 + r^3 + \ldots + r^{k-1} = \frac{r^k - 1}{r-1}.$$

Adding the next power of r to both sides of our assumption:

$$[1 + r + r^2 + \ldots + r^{k-1}] + r^k = \left[\frac{r^k-1}{r-1}\right] + r^k = \frac{r^k - 1 + r^{k+1} - r^k}{r-1} = \frac{r^{k+1}-1}{r-1}$$

Therefore, we have proven that if the identity is true for $n = k$, then it is also true for $n = k+1$. This concludes the inductive step, and the proof.

Problem 4.4. Let's calculate the first few terms of the new sequence: 2001, 2002, 2003, 2000, 2005, 1998, 2007, 1996, ... We can guess that

$$a_{2n-1} = 2n + 1999, \text{ and } a_{2n} = 2004 - 2n.$$

We see that this is true for $n = 1$. Now, the inductive step is proving that this is true for $n = k + 1$ given that it is true for $n = 1, 2, \ldots, k$. We have

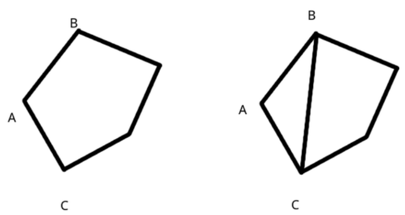
$$a_{2k+1} = a_{2k-2} + a_{2k-1} - a_{2k} = 2004 - 2(k-1) + 2k + 1999 - (2004 - 2k) = 2k + 2001,$$

$$a_{2k+2} = a_{2k-1} + a_{2k} - a_{2k+1} = 2k + 1999 + 2004 - 2k - (2k + 2001) = 2004 - 2(k+1).$$

So, this is true for all natural numbers n. So, we have $a_{2004} = 2004 - 2(1002) = \boxed{0}$.

Problem 4.5. Our base case here is for $n = 3$. Now, we assume that for some k, the sum of the interior angles of the convex k-sided polygon is $180(k-2)$ degrees. The inductive step would be to show that this implies that the sum of the interior angles of the convex $k+1$-sided polygon is $180((k+1)-2)$ degrees.

We see in the following diagram that when we add another side to a polygon, it is equivalent to attaching a triangle to the side.



This means that, since the sum of the interior angles of a triangle is 180 degrees, if a k-sided polygon's total is $180(k-2)$ degrees, a $k+1$-sided polygon's total must be $180(k-2) + 180 = 180((k+1)-2)$ degrees.

§5 Misleading Proofs

Problem 5.2. We cannot use WLOG here. The variables x, y, and z are not interchangeable. For example, $x + 2y + 3z$ is not equal to $z + 2y + 3x$ all the time. If we were able to use WLOG, the labels (here, the variables) have to be interchangeable.

Problem 5.3. This is an example of *circular reasoning*. In this proof, we assumed that our conclusion was true, and then used that assumption to prove our conclusion. Circular arguments are often logically valid, but they do nothing to help us make a proof. This circular proof is akin to a proof such as "It is raining. Therefore, it is raining," which does not actually prove anything.