Proof Techniques

Pleasanton Math Circle: Middle School

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§1 Warm-Up

Problem 1.1. Prove that if $x + y + z = 7$ and x, y, and z are distinct, positive integers, then one of these three numbers must be 4.

Problem 1.2. Prove that $\sqrt{2}$ is irrational.

Problem 1.3. Prove that there are an infinite number of primes.

Problem 1.4. Prove that $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$.

§2 WLOG

Definition 2.1. **Without loss of generality**, often abbreviated as WLOG, simply means that we choosing a specific case to solve that will make our solution easier, but the specific case does not matter. The proof of the specific case proves any case.

Let's talk about the solution to Problem 1.1. Now that you know what WLOG is, you can hopefully see where we can apply that here.

Solution. Assume WLOG that $x < y < z$. This is possible because the numbers are distinct and it doesn't matter at all which is which. For example, the difference between solutions $(1, 2, 4)$ and $(4, 1, 2)$ is not important in this problem, because they are just rearrangements of the same numbers.

If $x = 2$, then the sum $x + y + z$ is at least $2 + 3 + 3 = 9$, which is too big (this is because they are distinct integers). So, $x = 1$. Then, if $y = 3$, the sum is at least $1 + 3 + 4 = 8$, which is too big once again. So, y has to be 2, and that leaves $z = 4$. So, $(x, y, z) = (1, 2, 4)$ is the only set of numbers possible, and any solution to this problem is a rearrangement of that. Thus, one of these three numbers must be 4. □

Problem 2.2. How many ways seven people can sit around a circular table? Keep in mind that the table is circular!

§3 Contradiction

Theorem 3.1 (Proof by contradiction)
A method of establishing a proposition's truth by showing that assuming that the proposition is false leads to a contradiction.

Problem 3.2. Prove that if $a, b, c > 0$, then if $ax^2 + bx + c = 0$ has real solutions, both solutions are negative.

Problem 3.3. Is it true that in any group of 8 people, there are two people in the group who have the same number of friends among the people in the group?

§4 Induction

Theorem 4.1 (Principle of Induction)

Induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the base case) and that from each rung we can climb up to the next one (the step).

Remark 4.2. Important! Induction consists of two steps:

1. Prove $F(1)$ (knocking over the first domino).
2. Prove that $F(k)$ implies $F(k+1)$ for any positive integer k (proving that each domino falling knocks over the next).

Problem 4.3. Prove the geometric series formula for all positive integers n,

$$1 + r + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

Problem 4.4. In the sequence 2001, 2002, 2003, ... , each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence?

Problem 4.5. Prove that the sum of the interior angles of a convex n-sided polygon is $180(n-2)$ degrees. You can assume that the sum of the interior angles of a triangle is $180°$.

Remark 4.6. Proofs across positive integers, especially those that start at 0 or 1, are hints to try the method of induction.

§5 Misleading Proofs

Misleading proofs are often illustrations of *mathematical fallacies*. They use clever ways to disguise missteps in their reasoning, in order to come to a false conclusion. Examples of missteps include dividing by 0 or incorrectly taking square roots. Many times, they use one of the strategies we learned, but apply them incorrectly.

Example 5.1

Find the mistake in the following proof:
We wish to prove that every person in the world has the same eye color. We will do so using induction. Base case - in a group of one person, every person has the same eye color. Next - given that in any group of k people all have the same eye color, we will prove that a group with $k + 1$ people will also all have the same eye color. We do this as follows. Take a group of $k + 1$ people and remove one; the other k must all have the same eye color. Now, put back the removed person and remove another person; this leaves a group of k people, who all must have the same eye color. This includes the original removed person, so all $k + 1$ people must have the same eye color. This concludes the inductive step, so all people have the same eye color.

Solution. There is a problem in the inductive step here. When we have a group of two people, and we remove one, then the other person indeed has the same eye color as everyone else (just themselves). However, there is no one else in the group to have the same eye color to. Hence, the two people are not necessarily the same height, and the induction fails. $\square$

Problem 5.2. Find the mistake in the following proof:
Proof that if $x + 2y + 3z = 14$ and x, y, and z are distinct, positive integers, then one of these three numbers must be 2. We will assume WLOG that $x < y < z$. If $x = 2$, the sum is at least $2 + 2(3) + 3(4) = 20$. So, $x = 1$. Then, the sum is at least $1 + 2(2) + 3(3) = 14$, so this must be the only solution, and one of the numbers must be 2.

Problem 5.3. Find the mistake in the following proof:
Proof that if x is even x^2 is even. If x^2 is even, then it has to be divisible by 4, which means that x is divisible by 2. So, x is even, which means that x^2 must be even.