Constructions

Pleasanton Math Circle: Middle School

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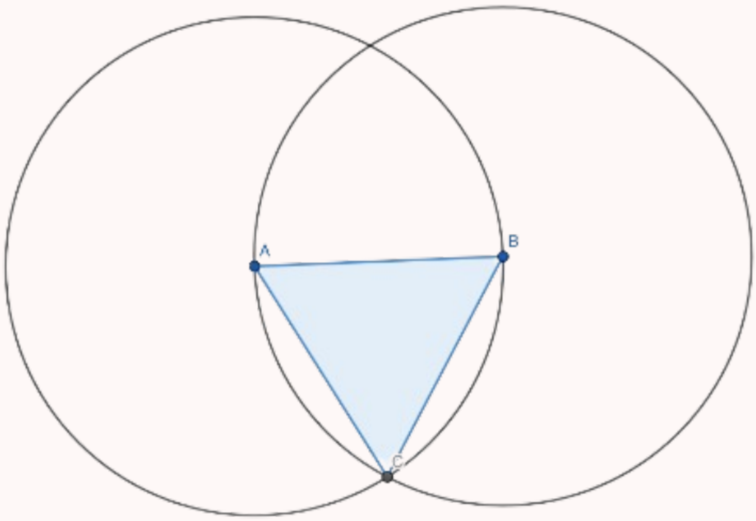
§1 Introduction

Definition 1.1. A geometric figure is considered constructible if it can be drawn perfectly with only a **straightedge** and **compass**.

A compass is useful because we can create circles that allow for equal length measure and straightedges allow us to draw lines between points.

Example 1.2

Let's say we wanted to construct an equilateral triangle. We can start by creating point A and point B. We can draw two circles centered at A through B and at B through A respectively. Let one of the intersections be C. Then ABC is equilateral.



Exercise 1.3. Prove that the above construction works.

§2 Common constructions

In this section, we'll be looking at some common constructions that are the building blocks of more complex ones. We've already seen how to construct an equilateral triangle, and the idea of trying to create equal length is going to be fruitful in the other examples as well.

For all of the following problems, starting might be challenging, so feel free to ask for hints.

Problem 2.1. Given two segments, construct a segment that has a length equal to the sum of the lengths of the two segments.

Problem 2.2. Given two segments, construct a segment that has a length equal to the difference of the lengths of the two segments.

Definition 2.3. For a segment AB, we say its perpendicular bisector is the line passing through the midpoint of the segment and it is perpendicular.

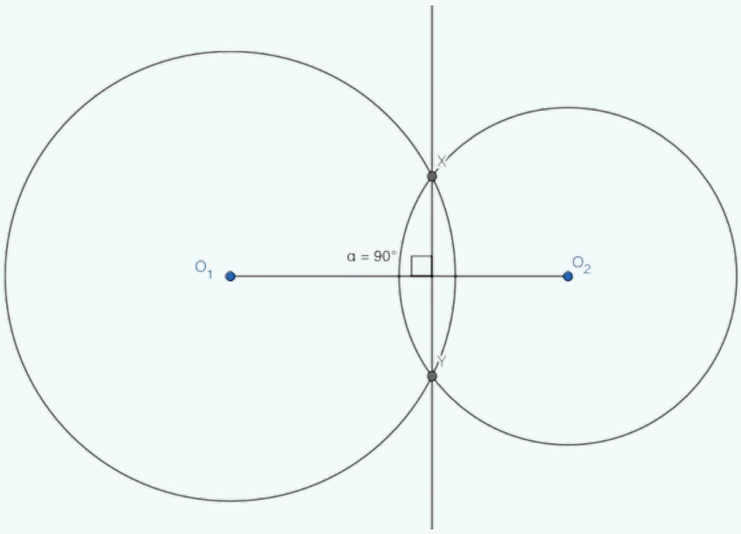
Example 2.4

If we consider the xy plane and the segment from $(-3, 0)$ to $(3, 0)$, the y-axis would be the perpendicular bisector, since it passes through the midpoint which is $(0, 0)$ and is perpendicular to the segment.

For the next few problems, the following theorem might be helpful.

Theorem 2.5

Let there be two circles centered at O_1 and O_2 and say they intersect at two points X and Y. Then the segment XY is perpendicular to the segment O_1O_2.



Problem 2.6. For a segment AB, find a way to construct the perpendicular bisector. This also allows us to construct the midpoint.

Problem 2.7. Given a segment AB and a point P on it, construct the line through P perpendicular to the segment. What if P is not on the segment?

Definition 2.8. Given three points A, B, C, the angle bisector of $\angle ABC$ is a line through B which splits the angle exactly in half. In other words, we want to find a point X such that $\angle XBA = \angle XBC$.

Problem 2.9. Given three points A, B, C, find a method to construct the angle bisector of $\angle ABC$. (Hint: The diagonals of a rhombus bisect the angles. You want to create a rhombus with one of the vertices at B.)

§3 Triangle centers

Given a triangle ABC, there are various centers of importance. We'll look at the 4 main ones and work on constructing them.

Definition 3.1. The circumcenter is the point O such that $OA = OB = OC$. It is the intersection of the perpendicular bisectors of AB, AC, and BC.

Problem 3.2. Find a way to construct the circumcenter.

Definition 3.3. The incenter I is the point equidistant from the sides of ABC. It is the intersection of the angle bisectors.

Problem 3.4. Find a way to construct the incenter.

Definition 3.5. The orthocenter H is the intersection of the altitudes of triangle ABC.

Problem 3.6. Find a way to construct the orthocenter.

Definition 3.7. The centroid G is the intersection of the medians of ABC.

Problem 3.8. Find a way to construct the centroid.

§4 Constructing regular polygons (optional)

In the start, we constructed an equilateral triangle, which is a regular polygon with 3 sides. This raises the natural question:

For which positive integers n can we construct a regular polygon with n sides?

Problem 4.1. Find a way to construct squares and hexagons ($n = 4$ and 6 respectively).

The full answer is much more complicated as we show here, but this is just for fun if you are curious.

Definition 4.2. Recall that a prime is any positive integer with two divisors. We say a prime is a Fermat prime if it can be expressed as $2^{2^n} + 1$ for a positive integer n.

The only known Fermat primes are $3, 5, 17, 257$, and 65537.

Theorem 4.3

A regular n-gon is constructible if and only if n can be represented as the product of some power of 2 and some subset of the Fermat primes.

For example, $n = 2^3 \cdot 17 \cdot 65537$ would be constructible. However, we can't use the same Fermat prime twice, so a number like $n = 3^2 = 9$ wouldn't work, which seems very counterintuitive.

The constructions get more and more complicated as the value of n increases, but we will show the difficulty of the construction for $n = 5$ to give you a sense.