Nuances in Probability — Solutions

Pleasanton Math Circle: Middle School

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§1 Warm-Up

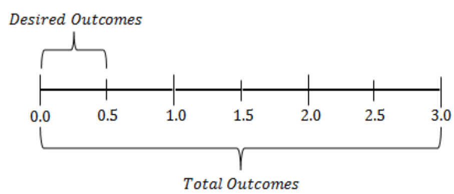
Problem 1.2. Since the result is even, there are three equally likely possibilities. Only one of these is 6, so the probability is $\boxed{\frac{1}{3}}$. This is the same type of probability question you are probably used to, just now, the total number of outcomes is being limited by conditions.

Problem 1.3. The outcome of the first 9 flips has no effect on the tenth flip, so the probability is still $\boxed{\frac{1}{2}}$, just like it would be on the first. This is known as the Gambler's fallacy.

Problem 1.4. We can use the application of $P(A|B) = \frac{P(A \cap B)}{P(B)}$ here. The probability that one of the children is a girl is $\frac{3}{4}$, because the possible combinations are (boy, boy), (girl, girl), (boy, girl), and (girl, boy). The probability that both are girls and one is a girl (or just that both are girls) is $\frac{1}{4}$. So, the probability that both are girls given that one is a girl is

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \boxed{\frac{1}{3}}.$$

Problem 1.5. We use a geometric approach with a number line from 0 to 3. It is clear that the desired numbers are those from 0 to 0.5.



We then measure lengths to calculate the probability, and we get

$$\frac{0.5}{3} = \boxed{\frac{1}{6}}.$$

Problem 1.6. See solution in section 4 of the handout.

§2 Conditional Probability

Problem 2.1. For the first question, we know that the first coin is heads, so we just need the probability that other two are heads, which is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

For the second question, we have a condition, but we don't know that a specific coin came up heads. It is a weaker condition to say that some coin was heads rather than a specific coin was heads. The probability is going to be

$$\frac{\text{Number of ways to have all heads}}{\text{Number of ways to have at least one head}} = \frac{1}{7}.$$

Problem 2.3. Here, we can calculate

$$\frac{\text{P(Person has the flu and tests positive)}}{\text{P(Person tests positive)}}.$$

The probability of having the flu and testing positive is $(0.05)(0.90) = 0.045$.

The probability of testing positive contains two cases: having the flu and correctly testing or being healthy and incorrectly testing. The probability is $(0.05)(0.90) + (0.95)(0.10) = 0.140$.

Therefore, the probability that a person who tests positive actually is infected is

$$\frac{0.045}{0.140} = \boxed{32.1\%}.$$

Problem 2.4. The key here is that Florida is a rare name. Let the probability that a child is a girl named Florida be $P(G_F) = a$ and $P(\text{girl who is not named Florida}) = P(G_{NF}) = \frac{1}{2} - a$. We have $P(B) = \frac{1}{2}$.

In a two child family with one girl named Florida, we have five possibilities: (B, G_F), (G_F, B), (G_F, G_{NF}), $(G_{NF}, G_F$, and (G_F, G_F).

We have $P(B, G_F) = P(G_F, B) = \frac{1}{2}a$, $P(G_F, G_{NF}) = P(G_{NF}, G_F) = a(\frac{1}{2} - a)$, and $(G_F, G_F) = a^2$.

The probability that both are girls is

$$2a(\frac{1}{2} - a) + a^2 = a - a^2.$$

The probability that one child is a girl named Florida is

$$2 \times \frac{1}{2}a + 2a(\frac{1}{2} - a) + a^2 = 2a - a^2.$$

Thus, the probability that both are girls given that one is a girl named Florida is

$$\frac{a - a^2}{2a - a^2} = \frac{1 - a}{2 - a} \approx \boxed{\frac{1}{2}}.$$

The answer is approximately $\frac{1}{2}$ because a is very small. The reason this condition matters is because there are three possibilities — (boy, girl), (girl, boy), and (girl, girl) — but the last one has a higher probability of having Floridas.

§3 The Monty Hall Problem

Problem 3.1. We went over this together, but here is a table that should explain this well.

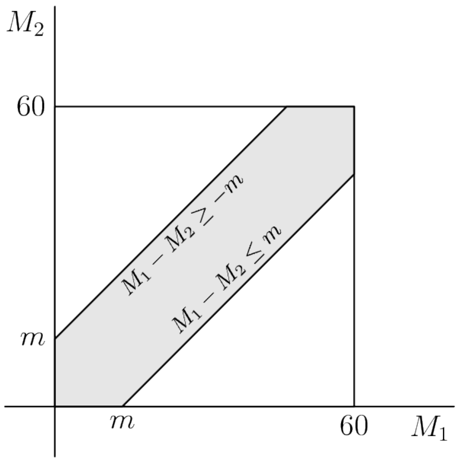
Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

§4 Geometric Probability

Problem 4.1. Since we have an infinite number of points in the circle, we can hardly count the desired and total outcomes. So, we use a geometric approach. We calculate the areas of the desired regions, and then divide those. So, the probability is

$$\frac{(\pi)(\frac{r}{2})^2}{(\pi)(r^2)} = \boxed{\frac{1}{4}}.$$

Problem 4.2. Let the two mathematicians be M_1 and M_2. Consider plotting the times that they are on break on a coordinate plane with one axis being the time M_1 arrives and the second axis being the time M_2 arrives (in minutes past 9 a.m.). The two mathematicians meet each other when $|M_1 - M_2| \leq m$. Also because the mathematicians arrive between 9 and 10, $0 \leq M_1, M_2 \leq 60$. Therefore, 60×60 square represents the possible arrival times of the mathematicians, while the shaded region represents the arrival times where they meet.



It's easier to compute the area of the unshaded region over the area of the total region, which is the probability that the mathematicians do not meet:
$\frac{(60-m)^2}{60^2} = .6$ $(60 - m)^2 = 36 \cdot 60$ $60 - m = 12\sqrt{15} \Rightarrow m = \boxed{60 - 12\sqrt{15}}$.