Graph Theory

Pleasanton Math Circle: Middle School

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§1 Introduction

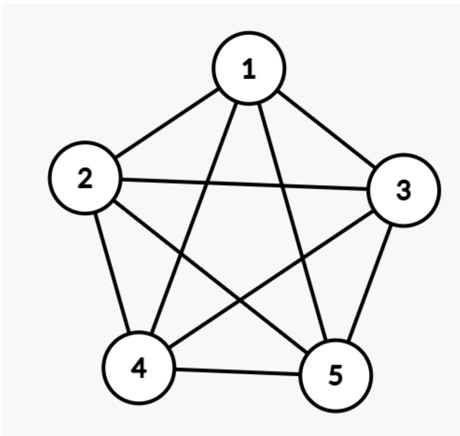
We'll start off with a lot of definitions so we can get into some more fun problems later.

Definition 1.1. A *node* is a vertex in the graph and is usually labelled with a number.

Definition 1.2. An *edge* connects two nodes, typically denoted with a line segment or curve that starts at one point and goes to the other. With this, the *degree* of a vertex is the number of edges going through it.

Definition 1.3. An *undirected graph* is a graph such that an edge is bidirectional, meaning that you can travel from one node to the other and vice versa. On the contrary, a *directed graph* is a graph such that nodes have a direction, usually denoted with an arrow, that signals that you can only travel from one of the nodes to the other and not the other way.

Problem 1.4. For the graph below, determine whether it is directed/undirected and the number of edges and nodes.



Problem 1.5 (Handshaking Lemma)**.** Prove that the sum of the degrees of all the nodes is even.

Problem 1.6. Given a graph with 8 nodes and degrees $1, 2, 3, 4, 5, 6, 7, x$, find x.

For all the following problems and definitions, we assume the graph is undirected (since I may have forgot to specify it everywhere).

§2 Walking on Graphs

A walk on a graph is a list of vertices such that every adjacent vertex is connected by an edge. A trail is a walk that never goes through the same edge twice and a path is a walk that never visits the same vertex twice. With these, we get the following definition:

Definition 2.1. A graph is cyclic if there exists a trail starting and ending at the same vertex, and it is called acyclic otherwise. A graph is connected if every node can be visited from any other node through a walk.

Problem 2.2. Prove that if an undirected graph is acyclic, at least one node has a degree of 0 or 1.

Definition 2.3. A connected component in a graph is the set of nodes and edges that are reachable through a walk from a single vertex. A graph can have multiple connected components.

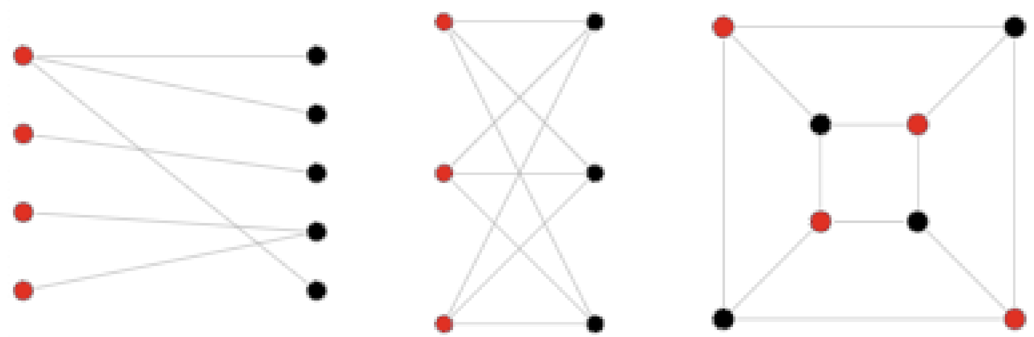
Problem 2.4. Prove that two distinct connected components (all the edges and nodes are not the same) do not share any nodes in a graph.

§3 Types of Graphs

Definition 3.1. A complete graph of size x (referred to as K_x) is a graph with x nodes such that every pair of nodes is connected with an edge. These are also referred to as $x-$cliques.

Problem 3.2. Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

Definition 3.3. A bipartite graph is a graph such that it is possible to split its vertices into two disjoint sets such that no two vertices in the same set are connected with an edge. Here are some examples:



Problem 3.4. Prove that all acyclic graphs are bipartite.

§4 Euler's Formula

First, we start off with some definitions.

Definition 4.1. A planar graph is one such that no edges intersect. Sometimes, a graph with intersecting edges can be redrawn in a way such that it becomes planar. However, sometimes it is impossible. The number of faces is the number of regions formed by the edges (including the outside).

Theorem 4.2 (Euler's Formula for Planar Graphs)

Given a planar graph with V vertices, E edges, and F faces, we have $V - E + F = 2$.

Problem 4.3 (USACO)**.** Farmer John decides to build a new fence around parts of his farm, but he keeps getting distracted and ends up building the fence into a much stranger shape than he intended! Specifically, FJ starts at position (0,0) and takes N steps, each moving one unit of distance north, south, east, or west. Each step he takes, he lays a unit of fence behind him. For example, if his first step is to the north, he adds a segment of fence from (0,0) to (0,1). FJ might re-visit points multiple times and he may even lay the same segment of fence multiple times. His fence might even cross over itself if his path cuts through a run of fencing he has already built.

Needless to say, FJ is rather dismayed at the result after he completes the fence. In particular, he notices that it may be the case that he has now partitioned off some areas of the farm from others, so that one can no longer walk from one region to another without crossing a fence. FJ would like to add gates to his fences to fix this problem. A gate can be added to any unit-length segment of fence he has built, allowing passage between the two sides of this segment.

Please determine the minimum number of gates FJ needs to build so that every region of the farm is once again reachable from every other region.

§5 Trees

Definition 5.1. We define a tree in 3 different ways, all of which imply the other 2 facts (which make trees very useful).

1. A connected graph with one less edge than the number of nodes.
2. A connected acyclic graph.
3. Every pair of nodes has a unique path between them.

With this, we get a nice result.

Theorem 5.2

The minimum number of edges in a connected graph with n nodes is $n - 1$, with equality achieved when the graph is a tree.

Problem 5.3. Prove that if you add an edge to a tree, a unique cycle is formed.