Extremal Principle

Pleasanton Math Circle: Middle School

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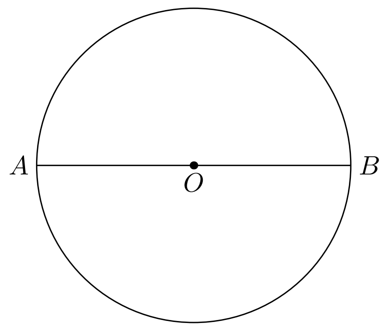
§1 Warm-Up

Let's first brush up on geometry knowledge that we will need for this problem.

Exercise 1.1. Consider a circle with diameter AB and center O.



Pick a point P on the circumference on the circle, and draw the line segments $\overline{AP}$ and $\overline{BP}$.

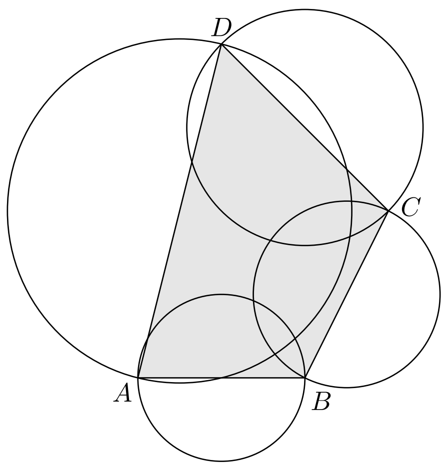
(a) Just by looking, what do you notice about the angle $\angle APB$? (If you want to prove this observation, then use the fact that AO, PO, and BO are equal lengths, then consider $\triangle APO$ and $\triangle BPO$).

(b) Let Q be some point inside the circle. What do you notice about the angle $\angle AQB$?

(c) Let R be some point outside the circle. What do you notice about the angle $\angle ARB$?

Use the observations you got from this exercise, and use it to solve this next problem.

Problem 1.2. Prove that for all possible convex quadrilaterals $ABCD$ (convex means all of the interior angles are less than 180°), the four circles with diameters of AB, BC, CD, and DA will cover the entire quadrilateral. Below is an example of one such $ABCD$.



§2 Definition

The **extremal principle** is a useful problem-solving tactic, broadly defined as focusing on a certain example with extreme properties, usually the smallest or largest of a certain thing. In Problem 1.2, we see how it's useful to think of the largest angle. In similar problems, we may think of focusing on the largest or smallest area, angle, side length, number, etc.

§3 Practice

Problems are arranged in roughly increasing order of difficulty.

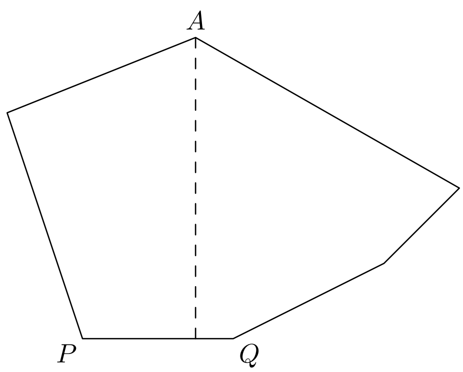
Problem 3.1. We are given four points in plane. Prove that you can always find three of those points that do not form an acute triangle.

Problem 3.2. Consider an infinite chessboard, the squares of which have been filled with positive integers. Each of these integers is the arithmetic mean of four of its neighbors (above, below, left, right). Show that all the integers are equal to each other.

Problem 3.3. Assume the earth is flat. There are a finite number of airports on the earth. At a certain moment in time, an airplane departs from each airport and flies to the nearest airport. Assume that the airports are located in such a way that no airport has two equally nearest airports. Prove that there is no airport to which more than five airplanes arrive.

Problem 3.4. Using the same conditions from Problem 3.3, prove that if there are an odd number of airports, then there will be an airport which no airplane flies to.

Problem 3.5. Consider any convex polygon where no two sides have equal lengths. Prove that you can always find three distinct vertices A, P, and Q, where PQ is a side of the polygon, such that the perpendicular from A to the line PQ intersects the segment PQ.



Problem 3.6. The numbers from 1 to n^2 are placed on a $n \times n$ grid (with no repeats). Prove there exists two cells that are vertically, diagonally, or horizontally adjacent such that the numbers in the cells differ by at least $n+1$.

Problem 3.7. In plane, there are n points. The area of any triangle with vertices in these points does not exceed 1. Prove that there exists a triangle with an area of 4 that contains all n points.

Problem 3.8. Prove that any convex polygon of area 1 can be placed inside a rectangle of area 2.