Finite Geometry

Pleasanton Math Circle: Middle School

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§1 Introduction

We have Euclidean Geometry, with an infinite number of points and lines. But what if we restrict ourselves to a finite number of points? The results of doing this are fun to play around with and have stumped mathematicians for decades.

§1.1 Rules

We can't just throw around points and a few lines and get anything fun from this. Let's add some definitions and rules.

Definition 1.1. A line is just a set of points. In the actual diagram, they don't have to be straight or anything like that. For example, if we have points $1, 2, 3$, we can define a line to be $\{1, 2, 3\}$.

We also don't want our geometry to be too basic.

- There exist at least 4 points such that no 3 of them belong to the same line.

These two rules make the geometry interesting.

- Given any two points, there is a unique line through them.
- This rule has two versions. We will refer to them as version 1 and 2. The first version: Given any two lines, they intersect at exactly 1 point. The second version: Given a line and a point not on that line, there exists exactly one line through that point such that the two lines don't share any points in common.

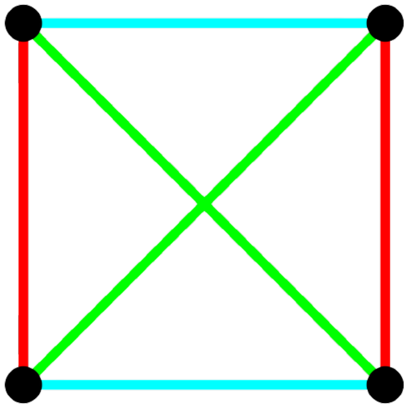
§2 Finite affine planes

These are the planes which satisfy version 2 of the second rule.

Problem 2.1. Explore! Find some affine planes. (in this case, some=1)

Proof.

Here is an affine with 4 points, 6 lines, and 2 points on each line:



$\square$

For the sake of time, from this point on, we only care about planes such that the number of points on each line is equal. Let there be k points and n points on each line. We want to find k in terms of n.

Problem 2.2. Find the number of lines in two different ways and use that to solve for k in terms of n.
Hint: For the first way, consider choosing two points. For the second way, first find the number of lines that each point lies on. Then use a combination of both rules to express the number of lines.

Proof. Since a pair of points determines a unique line, the total number of pairs of points is $\begin{pmatrix} n \\ 2 \end{pmatrix} \cdot x$ where x is the number of lines. However, the total numbers of pairs of points is also $\begin{pmatrix} k \\ 2 \end{pmatrix}$, so one way to represent the number of lines is $\frac{\begin{pmatrix} k \\ 2 \end{pmatrix}}{\begin{pmatrix} n \\ 2 \end{pmatrix}} = \frac{k(k-1)}{n(n-1)}$. For the second way, we first compute the number of lines each point lies on. There are $k-1$ points not counting the one point we excluded and to complete the line, we need $n-1$ more points. So each point lies on $\frac{k-1}{n-1}$ lines. Consider one line. For each point on the line, it lies on $\frac{k-1}{n-1} - 1$ lines other than that one. However, there are also $\frac{k-n}{n}$ lines parallel to that one. Then the total number of lines is

$$1 + n\left(\frac{k-1}{n-1} - 1\right) + \frac{k-n}{n}.$$

We can set this equal to $\frac{k(k-1)}{n(n-1)}$ and do some algebra to get that $k = n^2$. $\square$

Problem 2.3. Using your findings above, given an affine plane with n points on each line, find each of the following:

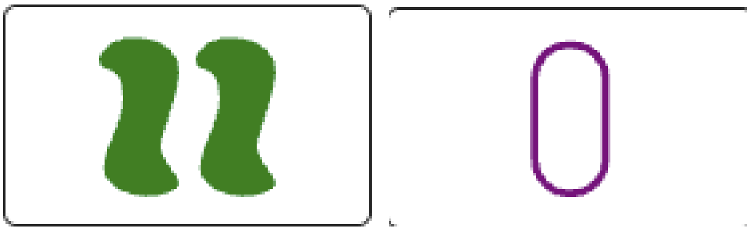
- The number of lines
- The number of points
- The number of lines each point lies on

This is called a Finite Affine plane of Order n.

Proof. We already found that the number of points is n^2.. We also found that the number of lines was $\frac{n^2(n^2-1)}{n(n-1)} = n^2 + n$. And lastly, the number of lines each point lies on is $\frac{n^2-1}{n-1} = n + 1$. $\square$

§3 Set

Set is a card game with 81 cards. Each card has 4 properties - a color, a number, a shape, a filling and there are 3 possibilities for each property. The colors are purple, red, green, the shapes are squiggles, ovals, diamonds, the fillings are empty, striped, filled, and the numbers are 1,2,3. Here are some examples:



The first is 2 filled green squiggles and the second is 1 empty purple oval. For the sake of simplicity, we will refer to these as 2FGS and 1EPO (the order is number, shading, color, and shape).

Definition 3.1. A set is 3 cards such that for each property, all the cards have the same value or all distinct values. For example 1FGS, 2FGO, and 3FGD are a set.

Problem 3.2. Given two cards, prove there is a unique third card that forms a set with the first 2.

Proof. We go through each property 1 by 1. Consider the color. If the first two cards have the same color, the third must have the same color. If they have different colors, then the third must have the 3rd color that is unused by both of the first cards. We can do the same for each property and see that it is uniquely determined, so the card is uniquely determined. □

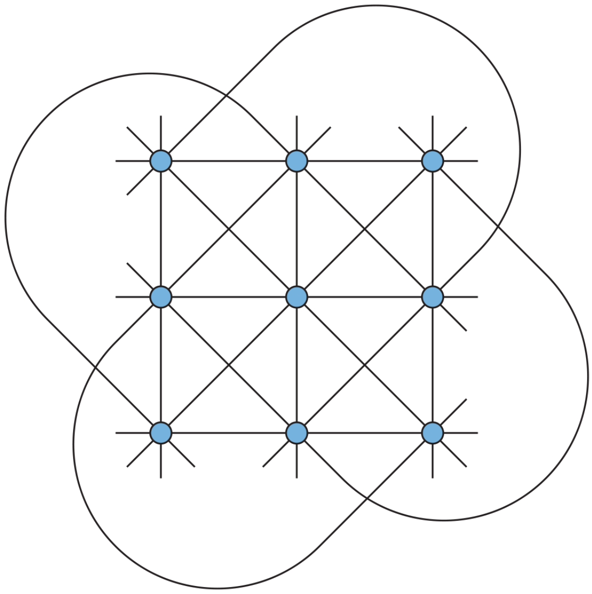
Problem 3.3. Using this information, what is the MAXIMUM number of sets we can form given 9 cards? Be careful, you need to divide by something!

Proof. It would be optimal that for each two cards we pick, there is a third card on the board that completes the set. This means there should be $\binom{9}{2}$ sets. But we need to divide by 3, since each set is counted is $\binom{3}{2} = 3$ times. This gives us 12. □

And now, the finale...

Problem 3.4. Find 9 cards that achieve this maximum (i.e. if you found that the maximum was n, find 9 cards that have n sets in them). Once you do this, let these cards be points and form "lines" through the ones that are part of a set (there are 3 points on each line). Do you notice something?

Proof. Here is the diagram you get with the points denoting cards and the sets denoting lines:



There are 3 points per line, 12 lines (sets), 9 points, and 4 lines through each point. □

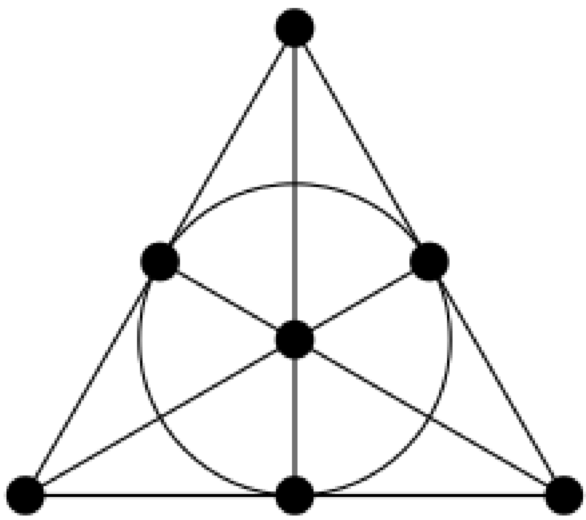
> **Remark 3.5.** This diagram is known as the Hesse Configuration if you want to research more about it.

§4 Finite Projective Planes

Finite projective planes satisfy version 1 of rule 2.

Problem 4.1. Find a finite projective plane!

Proof. This is called the Fano Plane:



□

From here, we will only work with the planes such that the number of points on each line is the same. Let there be $n+1$ points on each line.

Problem 4.2. Find the number of points in terms of n.
Hint: Count the number of pairs of points in two ways.

Proof. Let k be the number of points. One way to count the number of pairs of points is $\binom{k}{2}$. Proving this statement is quite time consuming, so we assume that the number of lines and points is the same. Then the other way to count is the number of lines times $\binom{n+1}{2}$, so $k\binom{n+1}{2} = \binom{k}{2}$ and using algebra, we get $k = n^2 + n + 1$. $\square$

Problem 4.3. Using your findings above, given a projective plane with $n+1$ points on each line, find each of the following:

- The number of lines
- The number of points
- The number of lines each point lies on

This is called a Finite Projective plane of Order n.

Proof. We assumed that the number of lines and points is the same, so we get $n^2 + n + 1$ lines and points. Consider one point and the other $n^2 + n$. Each line through that point contains n other points, so the number of lines that go through the point is $\frac{n^2+n}{n} = n + 1$. $\square$

Remark 4.4. It is worth noting that lines and points are quite similar in this type of finite geometry; they have a form of "duality". If you switch the words line and point in any statement we made, it will still hold true. The number of lines through each point is equal to the number of points on each line and the number of lines is equal to the number of points.

§5 Connecting!

This is a decently hard problem. It is inspired by the fact that there are actually not finite geometries for all orders, only some. It is conjectured (not proven yet!) that these finite geometries only exist if the order is a prime or a power of a prime.

Problem 5.1. Prove that if a finite affine plane of order n exists, then so does a finite projective plane of order n.

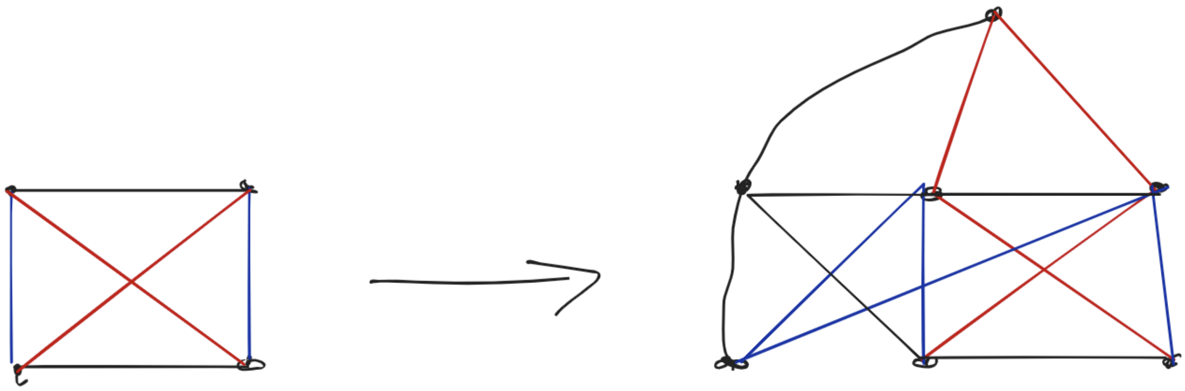
Proof. First, we define some terms.

Definition 5.2. A "parallel class" in a finite affine plane is a set of lines that are don't intersect (remember, they don't have to look parallel!).

Before we start on the proof, we first take note of the things we *want* to happen.

1. We need to add $n+1$ points.
2. We can only add ONE line.
3. We don't want any pair of lines that don't intersect.
4. We want to add one point to each of the existing lines.

We first notice that the number of lines in each parallel class is $\frac{n^2}{n} = n$ since we can divide the n^2 points into n lines of n points such that no two intersect. Now we start with some easy things: We add the $n+1$ points and make a new line through them. This completes Agenda Items 1,2. Now 3,4 are a bit trickier. Since each parallel class has n lines, there must be $\frac{n^2+n}{n} = n+1$ classes. Wait, we added $n+1$ points. Is that a coincidence? It is actually exactly what we need. If we take each of the parallel classes, we can extend each line in that class and make it go through one of the new points. This means every pair of lines that were parallel now intersect, just as we would like to happen. We can see that the last agenda item is automatically completed when we do this, so we are done! Here is an illustrative example (each color denotes a parallel class):



$\square$