Finite Geometry

Pleasanton Math Circle: Middle School

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§1 Introduction

We have Euclidean Geometry, with an infinite number of points and lines. But what if we restrict ourselves to a finite number of points? The results of doing this are fun to play around with and have stumped mathematicians for decades.

§1.1 Rules

We can't just throw around points and a few lines and get anything fun from this. Let's add some definitions and rules.

Definition 1.1. A line is just a set of points. In the actual diagram, they don't have to be straight or anything like that. For example, if we have points $1, 2, 3$, we can define a line to be $\{1, 2, 3\}$.

We also don't want our geometry to be too basic.

- There exist at least 4 points such that no 3 of them belong to the same line.

These two rules make the geometry interesting.

- Given any two points, there is a unique line through them.
- This rule has two versions. We will refer to them as version 1 and 2. The first version: Given any two lines, they intersect at exactly 1 point. The second version: Given a line and a point not on that line, there exists exactly one line through that point such that the two lines don't share any points in common.

§2 Finite affine planes

These are the planes which satisfy version 2 of the second rule.

Problem 2.1. Explore! Find some affine planes. (in this case, some=1)

For the sake of time, from this point on, we only care about planes such that the number of points on each line is equal. Let there be k points and n points on each line. We want to find k in terms of n.

Problem 2.2. Find the number of lines in two different ways and use that to solve for k in terms of n.
Hint: For the first way, consider choosing two points. For the second way, first find the number of lines that each point lies on. Then use a combination of both rules to express the number of lines.

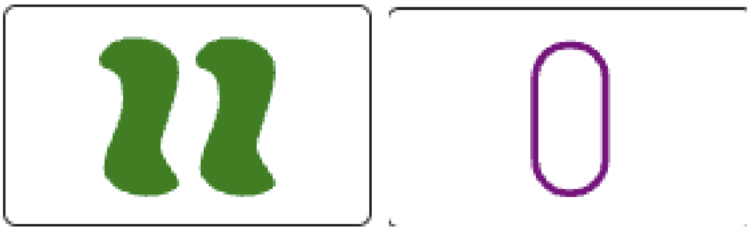
Problem 2.3. Using your findings above, given an affine plane with n points on each line, find each of the following:

- The number of lines
- The number of points
- The number of lines each point lies on

This is called a Finite Affine plane of Order n.

§3 Set

Set is a card game with 81 cards. Each card has 4 properties - a color, a number, a shape, a filling and there are 3 possibilities for each property. The colors are purple, red, green, the shapes are squiggles, ovals, diamonds, the fillings are empty, striped, filled, and the numbers are 1,2,3. Here are some examples:



The first is 2 filled green squiggles and the second is 1 empty purple oval. For the sake of simplicity, we will refer to these as 2FGS and 1EPO (the order is number, shading, color, and shape).

Definition 3.1. A set is 3 cards such that for each property, all the cards have the same value or all distinct values. For example 1FGS, 2FGO, and 3FGD are a set.

Problem 3.2. Given two cards, prove there is a unique third card that forms a set with the first 2.

Problem 3.3. Using this information, what is the MAXIMUM number of sets we can form given 9 cards? Be careful, you need to divide by something!

And now, the finale...

Problem 3.4. Find 9 cards that achieve this maximum (i.e. if you found that the maximum was n, find 9 cards that have n sets in them). Once you do this, let these cards be points and form "lines" through the ones that are part of a set (there are 3 points on each line). Do you notice something?

§4 Finite Projective Planes

Finite projective planes satisfy version 1 of rule 2.

Problem 4.1. Find a finite projective plane!

From here, we will only work with the planes such that the number of points on each line is the same. Let there be $n + 1$ points on each line.

Problem 4.2. Find the number of points in terms of n.
Hint: Count the number of pairs of points in two ways.

Problem 4.3. Using your findings above, given a projective plane with $n + 1$ points on each line, find each of the following:

- The number of lines
- The number of points
- The number of lines each point lies on

This is called a Finite Projective plane of Order n.

§5 Connecting!

This is a decently hard problem. It is inspired by the fact that there are actually not finite geometries for all orders, only some. It is conjectured (not proven yet!) that these finite geometries only exist if the order is a prime or a power of a prime.

Problem 5.1. Prove that if a finite affine plane of order n exists, then so does a finite projective plane of order n.