Guarding a Base

Pleasanton Math Circle: Middle School

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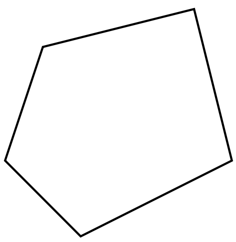
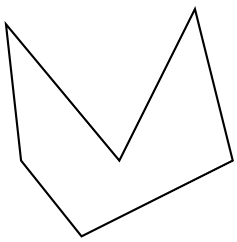
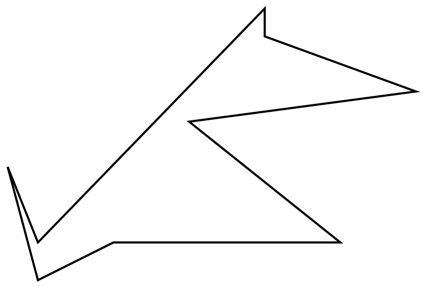
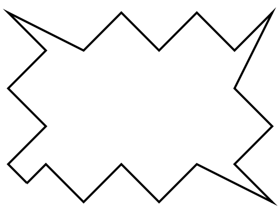
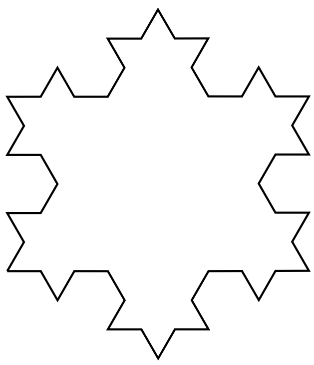
§1 Introduction

You are the commander of a new military base (which is a polygon). You remember that you must hire guards to defend the base. To completely defend the base, the guards must be able to see every part of the building. Additionally, since the government has a shortage of money, you need to hire the minimum number of guards. The guards can stand anywhere inside the building and can see in all directions, but cannot move. What is the minimum number of guards that you must hire?

§2 Warm Up

How many guards will we need to defend these bases?

§3 Questions to consider

1. In a convex (all angles less than 180) military base, how many guards are necessary to supervise the entire base?

2. In the base furthest to the right, show that you have optimally guarded the military base. (Show that you can't do it with any fewer guards.)

3. Find a base that needs 3 guards. What is the fewest number of sides such a base has?

4. What would happen if the base has 5 sides? (Try drawing different pentagons.)

5. Is it better for the guard to be on the sides of the polygon, or inside the polygon?

6. Prove that you can always guard an n-sided polygon with n guards. Can you improve this bound? (Can you replace n with something lower, like $n-1$?)

§4 Theorem

This problem was first proposed in 1973 by Victor Klee and became known as the Art Gallery Problem. The answer to the question can be found in the following theorem. The first proof was given in 1975 by Vasek Chvatal, but the most famous (and beautiful) proof was produced in 1978 by Steve Fisk. In the following sections, you will reproduce his proof.

Theorem 4.1 (Art Gallery Theorem)
Only $\lfloor \frac{n}{3} \rfloor$ guards or fewer are necessary to guard a n-sided polygon. This is also the best possible bound.

The floor function $\lfloor x \rfloor$ here denotes the largest integer less than or equal to x. For example, $\lfloor 3.321 \rfloor = 3$ and $\lfloor 11.482 \rfloor = 11$.

Exercise 4.2. Compute $\lfloor \sqrt{53} \rfloor$, $\lfloor 3.0 \rfloor$, and $\lfloor 728.3028 \rfloor$

The first part of the theorem states that given an n-sided polygon, $\lfloor \frac{n}{3} \rfloor$ is always enough. For example, for any given decagon (10 sided polygon), $\lfloor \frac{10}{3} \rfloor$ guards will be sufficient.

Exercise 4.3. Given that the first part of the theorem holds, how many guards do we need for a 29 sided polygon?

We will prove this part of the theorem second. The second part of the theorem states that this is the best possible bound, meaning that any number of guards lower than $\lfloor \frac{n}{3} \rfloor$ isn't enough to guarantee that they will be able to guard any n sided polygon. This is equivalent to the following proposition.

Proposition 4.4
There exists an n-sided polygon that requires $\lfloor \frac{n}{3} \rfloor$ guards.

Now we will show that this is true.

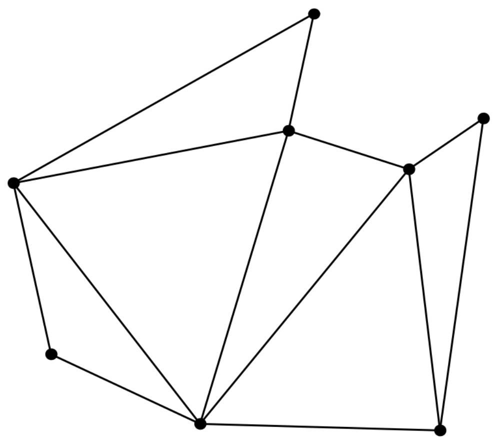


Figure 1: A triangulation

Exercise 4.5. Find a hexagon that requires 2 guards and a 9-sided polygon that requires 3 guards.

Exercise 4.6. Find a general construction of a $3n$-gon that requires n guards. (Hint: Try extending your constructions in the previous exercise)

Problem 4.7. Prove **Proposition 4.4**

Now that we proved part two of the theorem, lets get back to the first part. First, we need a quick definition.

Definition 4.8. A **triangulation** of a polygon is made by using diagonals to divide the polygon into triangles like in the figure above.

Lemma 4.9

Any triangulation of a polygon has a *nice 3-coloring*.

This just means that we can color the vertices of a polygon with three colors such that every triangle has all three colors. Equivalently, no vertices that are connected are colored the same. For an example, see the figure below.

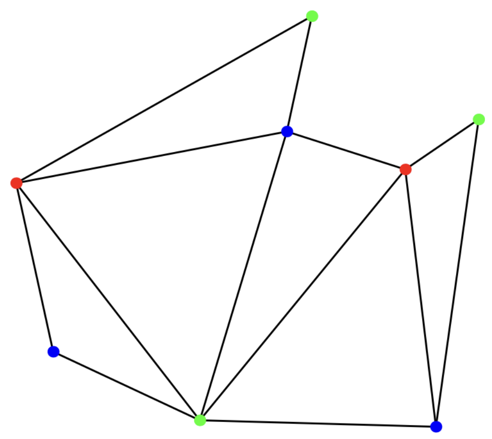


Figure 2: A nice 3-coloring

Exercise 4.10. Prove Lemma 9. First see that a triangle has a nice 3-coloring. Now using this, can you show tat a quadrilateral has a nice 3-coloring. From this, can we show that if an n-gon has a nice 3-coloring, then a $n+1$ gon also has a nice-3 coloring? Why does this show that all n-gons have a nice 3-coloring?

Exercise 4.11. Prove the Art Gallery Theorem using **Lemma 4.9**.