Path Walking

Pleasanton Math Circle: Middle School

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§1 Warm-Up

Problem 1.1. We start on the square marked S. Every minute, we move to the square above or the square to the right of where we are, as long as we do not leave the 3×3 board. How many ways can we travel from S to E?

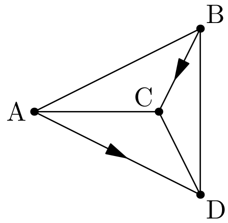
		E
S		

Problem 1.2. Consider the same rules as Problem 1.1, except now we cannot travel to the square marked X. Now how many ways can we travel from S to E?

	X	E
S		

Problem 1.3. A frog starts on the number 0 on a number line. Every minute, the frog can jump to either the next smallest multiple of 3 or the next smallest multiple of 5. The frog stops once it reaches 15. An example of a possible jump pattern is 0-3-5-6-10-15. How many possible ways can the frog jump from 0 to 15?

Problem 1.4. This diagram below shows four points A, B, C, and D, with the edges connecting them. During each move, you may travel to another point along a given edge. An arrow drawn on an edge means that you may only travel in the indicated direction; no arrow on an edge means that you may travel in either direction. You start at point A. If you make exactly 4 moves, how many such paths will end at A?



§2 One-Way Paths

Let's start off with a solution to Problem 1.1. It's already a simple problem, and this solution may seem unnecessarily complicated. However, in these next few problems we will see how this strategy of solving one-way paths will make things easier.

Solution. Because we start at square S, we will mark that square with a "1", which represents the number of ways we can get to that square. When you think about it, there is only one sequence of moves that gets you from S to S: doing nothing. In order to find the number of another square N, we identify the squares that lead directly to the N and take the sum of all their numbers. Here is what the completed grid will look like:

1	3	6
1	2	3
1	1	1

The number of the square E is 6, so there are $\boxed{6}$ ways to get from S to E.

Why does this strategy work? We will discuss this in detail in class, but the main idea is that in this problem, we know exactly how we can get to each position on the board. If we know which squares lead to square E on the next move, then we need to figure out how many ways we can get to the squares that get us directly to E. □

Problem 2.1. A chess king starts on the bottom-left square of a 5×5 grid. Each move, the king can move either a square up, a square to the right, or a square diagonally up-right. In how many ways can the king move from the bottom-left to the top-right square?

Problem 2.2. I start on the square marked S and I want to move to the square marked E by either moving up or to the right each minute. However, I MUST pass through the point marked X. How many possible paths could I take?

						E
			X			
S						

Problem 2.3. A frog starts on the number 0 on a number line. Every minute, the frog can jump to either the next smallest multiple of 5 or the next smallest multiple of 7. How many possible ways can the frog jump from 0 to 35?

Problem 2.4. You start on (0, 0) on a coordinate plane. Each minute you can either move one unit upward or one unit right. However, your y-coordinate may never be greater than your x-coordinate. Given these restrictions, how many ways can you travel from (0, 0) to (5, 5)?

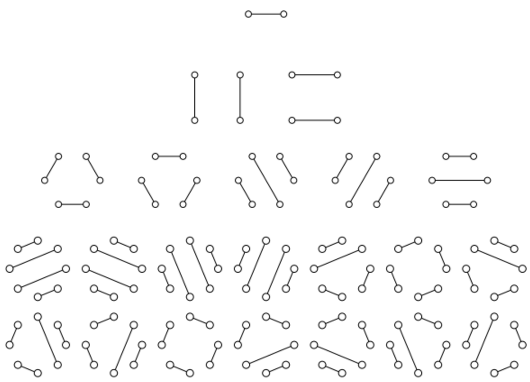
Problem 2.5. Ever notice what happens when you have a ton of parentheses in a math expression? It can get messy sometimes, but the way the parentheses are written should always be correct. Take for example the expression $((3 + 2) \cdot 4) \div (2 - 4)$. This example makes sense to compute. On the other hand, an expression like $(3 + 2) \cdot 4) \div (2 + (-4)$ does not work.

Now let's consider the "valid" and "invalid" ways to write parentheses. The first example has parentheses like this: (())(), and this is valid. The second example has parentheses like this: ())((), and this is invalid.

(a) We write n open-parentheses and n close-parentheses in a row in some order. Observe the differences between the valid arrangements and invalid arrangements. What makes an arrangement valid?

(b) How many different valid arrangements of 5 open-parentheses and 5 close-parentheses are there?

Problem 2.6. $2n$ people sit around a circular table. These people shake hands with each at the same time, so there will be n total handshakes among pairs of people. Assume that people's arms are long enough to reach anyone else at the table. Also, no handshake will cross another pair's handshake. In the following diagram, there are examples of possible handshake configurations if there are 2, 4, 6, or 8 people at the table:



How many total ways can 10 people sitting at this table shake hands?

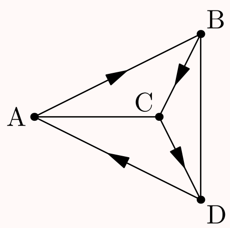
§3 Dynamic Programming Tables, kinda

This is only kinda because dynamic programming is really a computer programming topic, which I have no knowledge about. However, its application to math problems is still useful.

Dynamic programming is another strategy we can use to keep track of the number of ways to get from one place to another in a certain number of moves. We use dynamic programming when it's possible to return to places we've already been; in other words, it is not a one-way path like before. Here is an example of how we can use tables to help us solve such problems.

Example 3.1

This diagram below shows four points A, B, C, and D, with the paths connecting them. During each move, you may travel to another point along a given path. An arrow drawn on a path means that you may only travel in the indicated direction; no arrow on a path means that you may travel in either direction. You start at point A. If you make exactly 8 moves, how many such paths will end at A?



Solution. First, write down all possible paths you can take. This is to make it easier to work with, so you won't have to refer to the diagram. On the left column of the table will be the endpoint, and the right column will be all starting points that can lead to that endpoint.

Point	Can be reached from...
A	C, D
B	A, D
C	A, B
D	B, C

For example, $C \to A$ and $D \to A$ are legal moves according to the diagram, which is how the first row is made. Now we can make a dp table. This is how it starts off:

n	A	B	C	D
0	1	0	0	0
1				
2				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In out table, n is the number of moves. The "1" in column A row $n = 0$ means that after 0 moves (basically the starting position), there is one way to get to A. Define this value as $A(0)$. $B(1)$, meaning the number of ways to get to B after 1 move, can be found like this: reference the first chart we made. We can get to B from either A or D, so $B(1) = A(0) + C(0)$ (or more generally, $B(n) = A(n-1) + C(n-1)$). This is true because from A or C, we can make another move to get to point B. $B(1) = A(0) + C(0) = 1 + 0 = 1$.

We can fill out following rows by finding which values lead to the that point, then add up the sums of the values in the previous row. Here is what the table looks like up to $n = 8$:

n	A	B	C	D
0	1	0	0	0
1	0	1	1	0
2	1	0	1	2
3	3	3	1	1
4	2	4	6	4
5	10	6	6	10
6	16	20	16	12
7	28	28	36	36
8	72	—	—	—

$A(8)$ represents the number of paths possible that end at A after 8 moves, which is what we wanted to solve. The answer is $\boxed{72}$. □

Problem 3.2. Refer to Problem 1.4. How many paths are there that start from A and end at A and take exactly 6 moves?

Problem 3.3. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

Problem 3.4. An ant starts on a vertex of a cube and walks from vertex to vertex along the edges of the cube. How many ways can the ant travel across 6 edges and end up at the same vertex it started from?