Path Walking — Solutions

Pleasanton Math Circle: Middle School

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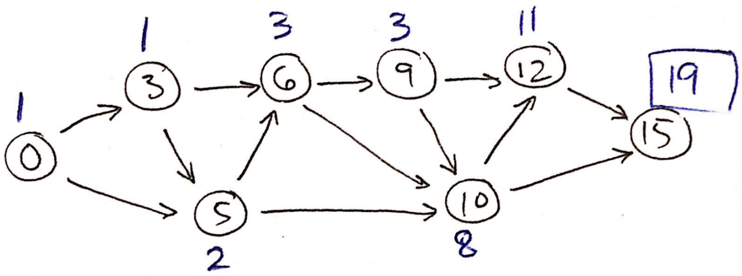
§1 Warm-Up

Problem 1.1. $\boxed{6}$. See solution on the second page of the notes handout.

Problem 1.2. $\boxed{3}$. Because you cannot travel on the X, the only way to get to E is from below it. Here's what the grid looks like when we will in the numbers. You can consider square X to have the number 0.

1	X	3
1	2	3
1	1	1

Problem 1.3. $\boxed{19}$. Here's a map of the paths the frog can go.



Problem 1.4. $\boxed{9}$. A more sophisticated way to solve this problem is shown in the solution to Problem 3.2. To solve this problem, however, we can also list out all possible paths that we can take from A back to A. Here are all possible paths of 4 moves:

- ABABA
- ABDBA
- ACACA
- ACDCA
- ABACA
- ACABA
- ABDCA
- ACDBA
- ADBCA

§2 One-Way Paths

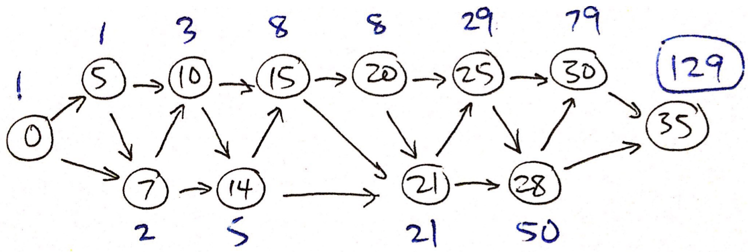
Problem 2.1. 321. The left column and bottom row of numbers are still all 1. Because the king is also allowed to move diagonally one space up-right, the numbers in all the remaining spaces are now the sum of the three numbers to the left, to the bottom, and to the bottom-left.

1	9	41	129	321
1	7	25	63	129
1	5	13	25	41
1	3	5	7	9
1	1	1	1	1

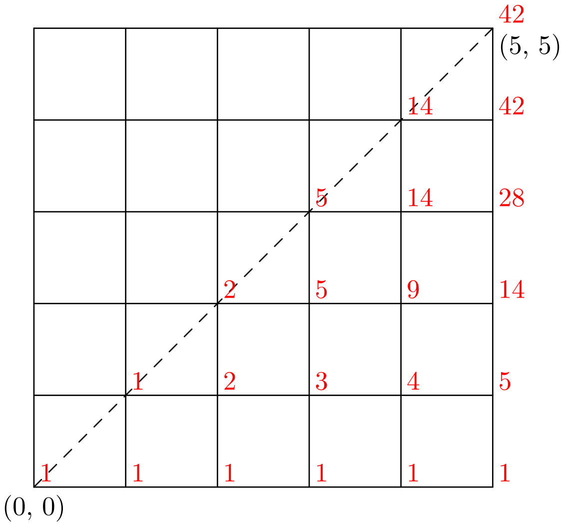
Problem 2.2. 40. You'll see that we can ignore the cluster of spaces on the top-right and bottom-left of the grid, because if I travel to those squares then I can never travel to the X during that path. Ignoring those, we can apply the same strategy to our grid like so:

—	—	—	10	20	30	40
1	3	6	10	10	10	10
1	2	3	4	—	—	—
1	1	1	1	—	—	—

Problem 2.3. 129.



Problem 2.4. 42. Here's the grid we use, and notice how we ignore any points above the dashed line in our solution.



Problem 2.5. (a) Imagine that you're writing this sequence of (and) from left to right. At no point when you're writing it should there be more) than (. For example, for the sequence ())((), if we write the first three parentheses we get ()), and there are more) than (, which tells us right away that one of those) does not have a (to match it. Therefore, the criteria for a valid sequence of parentheses is that we never have more) than (when we're writing this from left to right, and that at the end we have an equal number of (and) in the sequence.

(b) $\boxed{42}$. This is the same answer as problem 2.4. Imagine that a (in our sequence is code for moving one unit right in Problem 2.4, and that a) in our sequence is code for moving one unit up. After ten moves, we end up at (5, 5), and we ensure that we never have more) than (at any point, so these two problems are really just different ways to portray the same problem. Therefore, they have the same answer.

Problem 2.6. $\boxed{42}$. This answer is the same as 2.4 and 2.5 because it's also a different way to portray the same concept. This problem is too complex for me to explain clearly on paper. For now, if you would like to read more about this idea, look up Catalan numbers or read this paper: http://www.geometer.org/mathcircles/catalan.pdf. If you have lingering questions about this, please see me in person.

§3 Dynamic Programming Tables

Problem 3.1. $\boxed{50}$. Here are the two tables that should be made:

Point	Can be reached from...
A	B, C
B	A, D
C	A, B, D
D	A, B, C

n	A	B	C	D
0	1	0	0	0
1	0	1	1	1
2	2	1	2	2
3	3	4	5	5
4	9	8	12	12
5	20	21	29	29
6	50	—	—	—

Problem 3.2. $\boxed{351}$. This table shows the possible paths you can take from point to point:

Point	Can be reached from...
S	P_1, P_5
P_1	S, P_2
P_2	P_1, P_3
P_3	P_2
E	P_3, P_4
P_4	P_5
P_5	S, P_4

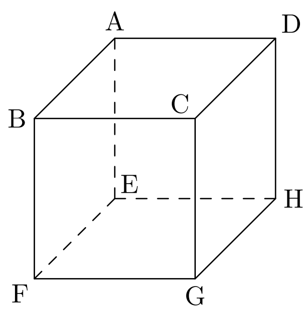
Notice that since the frog stops once it reaches E, that is why there are points that can lead to E, but E does not lead to any other point. Let n be the number of jumps the frog makes. In this solution, define $S(n), P_1(n), P_2(n), \ldots$ to mean the number of ways to get to that point in n jumps. In that case, the answer we want is $E(0) + E(1) + E(2) + \cdots + E(12)$, which is the number of ways to get to E in 12 moves or less.

Here is the dp table we make, which is similar to what we did for Example 6. Values of 0 are omitted.

n	P_4	P_5	S	P_1	P_2	P_3	E
0			1				
1		1		1			
2	1		2		1		
3		3		3		1	1
4	3		6		4		1
5		9		10		4	3
6	9		19		14		4
7		28		33		14	9
8	28		61		47		14
9		89		108		47	28
10	89		197		155		47
11		286		352		155	89
12	—	—	—	—	—	—	155

Adding up all values in the E column gives us $E(0) + E(1) + E(2) + \cdots + E(12)$, which is $\boxed{351}$.

Problem 3.3. $\boxed{182}$. Here is how we label the cube:



This table shows the possible paths. Notice that the three points that lead to any point are the three points that share an edge with it.

Point	Can be reached from...
A	B, D, E
B	A, C, F
C	B, D, G
D	A, C, H
E	A, F, H
F	B, E, G
G	C, F, H
H	D, E, G

Suppose we start at A. In this case, notice that B, D, and E would be symmetric around point A, so the values for those three points would always be the same. Same goes for C, F, and H. Knowing this fact would save you time filling out the dp table. Here is the finished dp table. Values of 0 are omitted:

n	A	B	C	D	E	F	G	H
0	1							
1		1		1	1			
2	3		2			2		2
3		7		7	7		6	
4	21		20			20		20
5		61		61	61		60	
6	182	—	—	—	—	—	—	—