Sizes of Infinities — Solutions

Pleasanton Math Circle: Middle School

Ryan Vir, Rohan Garg, Ryan Fu

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§1 Notation

The **cardinality** of a set of numbers is a fancy word for the size, or the number of elements in it. For a set S, the cardinality is referred to as $|S|$.

The symbols $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$ refer to the sets of integers, positive integers, rational numbers, and real numbers respectively.

§2 Warm Up

Problem 2.1. What is the cardinality of the set $\{1, 2, 3, \dots, 100\}$?

Proof. It's 100, since there are 100 elements. □

Problem 2.2. What do you think is larger: the cardinality of the positive integers or the cardinality of the integers?

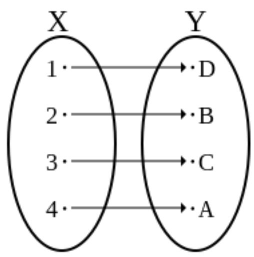
Proof. Intuitively, it seems like the cardinality of the integers would be higher, since it contains elements that the set of positive integers does not. But when comparing infinites, they are actually the same! We will see this later. □

§3 Comparing Infinite Sets

Say we wanted to prove two sets have the same cardinality. We can use a technique called a **one-to-one correspondence**.

The idea is to match up elements in the sets such that each element in the first set maps to a single element and each element in the second set is mapped to by a single element from the first set.

Let's say we wanted to prove $Y = \{A, B, C, D\}$ and $X = \{1, 2, 3, 4\}$ have the same cardinality. We can match elements as follows:



How do we do this for sets of infinite size? Let's demonstrate this with an example: the set of positive integers and the set of positive even integers.

Surprisingly, they have the same cardinality. Let's say we have a positive integer x, and we want to match it to an even integer. If we let x match with $2x$, is this a one-to-one correspondence?

Problem 3.1. Verify whether the matching stated above is a one-to-one correspondence or not.

Proof. Consider an even integer, $2x$. It is mapped to by exactly one integer from the domain, x. Similarly, each element in the domain is mapped to exactly one value in the range. □

It seems very uneasing that the set of positive integers has the same size as the set of positive even integers, since we don't count any of the odd numbers! But when dealing with infinities, this can happen pretty often.

Problem 3.2. Revise your answer to 2.2 and try to prove $|\mathbb{Z}| = |\mathbb{N}|$.

The exact correspondence is for a positive integer n, if it is odd, then send it to $-\frac{n-1}{2}$. If it is even, send it to $\frac{n}{2}$. It is easy to verify this works.

§4 More Challenging Correspondences

Problem 4.1. Prove that the set of real numbers x such that $0 < x < 1$ has the same cardinality as the set of all the real numbers $x > 1$.

Short, but a little harder to find: send x to $\frac{1}{x}$.

Problem 4.2. (Challenge) Prove that $|\mathbb{Q}| = |\mathbb{N}|$.

The idea is to sort the fractions by the sum of their numerator and denominator (ties can be broken arbitrarily). Then go down this list and label the fractions by the position in the list. Obviously, we label every fraction with a positive integer and every positive integer is used, so this works.

§5 Set of Reals

One of the biggest discoveries in mathematics was that actually, $|\mathbb{R}| \neq |\mathbb{N}|$! This is surprising since all these other infinities had the same size, but the real numbers just stood out.

The method used to prove this was Cantor's diagonal argument.

The idea behind it is to assume for contradiction that there does exist a valid mapping of the natural numbers to the real numbers, and proving that some real number is actually not mapped to. We will go over this in class with more details.