Paradoxes

Pleasanton Math Circle: Middle School

Ryan Vir, Rohan Garg, Ryan Fu

September 14, 2023

§1 Warm-up

We've talked about problem 1.1 last year, so we'll spend a minute or two on it right now.

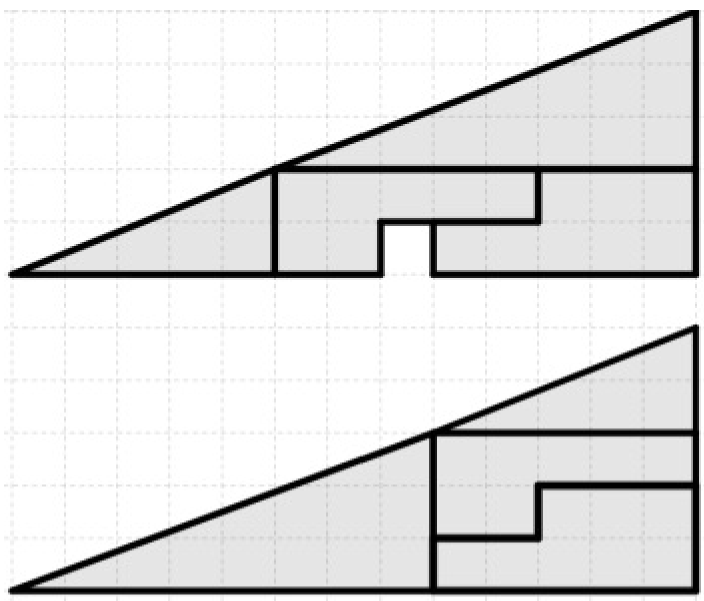
Problem 1.1. (Monty Hall problem) Suppose you are on a game show, and you have a chance to win a prize. You are given a choice of 3 doors: behind one is a brand new car, and there is nothing behind the other two. You initially pick one of the doors, and then the host, who knows what's behind the doors, opens one of the other two doors to reveal nothing behind that door. He then asks if you would like to switch your choice. Should you switch?

Problem 1.2. (Bertrand's Box) You are given 3 boxes. One contains 2 gold coins, one contains 2 silver coins, and one contains 1 gold and 1 silver coin. You randomly choose a box and randomly take a coin out. It's a gold coin! What is the probability that the other coin in the box is gold?

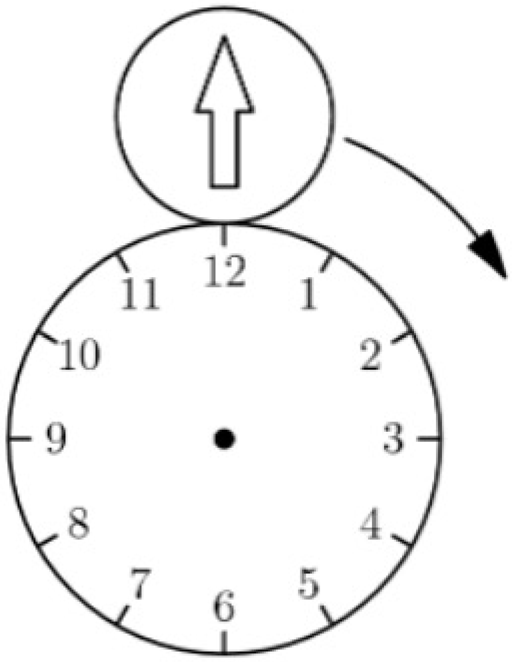
§2 Geometry

The following are paradoxes involving geometric figures:

Problem 2.1. (The missing square) The following two figures are 13×5 right triangles formed by using the same four pieces, but one contains a 1×1 hole in it. How can this be?



Problem 2.2. (2015 AMC 10A) The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



§3 Harder Problems

Problem 3.1. (Bertrand's other paradox) Consider an equilateral triangle inscribed in a circle. What is the probability that a randomly chosen chord of the circle is longer than a side of the triangle?

Problem 3.2. (100 prisoners) The problem goes: "The director of a prison offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategy — but may not communicate once the first prisoner enters to look in the drawers. What is the prisoners' best strategy?" Is there a way to beat $(\frac{1}{2})^{100}$?