Continued Fractions

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§1 Introduction

Definition 1.1. A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{k-1+\cfrac{1}{a_k}}}}}}$$

where $a_0, a_1, \ldots a_k$ are natural numbers. A continued fraction of the above form can be denoted as $[a_0, a_1, \ldots a_k]$ for short.

Exercise 1.2. Simplify each of the following continued fractions:

1. $[2, 3, 2]$
2. $[1, 4, 3, 4]$
3. $[6, 9, 4, 2]$
4. $[9, 12, 21, 2]$

Exercise 1.3. Write each of the following as a continued fraction:

1. $5/12$
2. $5/3$
3. $33/23$
4. $37/31$

Exercise 1.4. Find all positive integer solutions to $x + \cfrac{1}{y + \cfrac{1}{z}} = \frac{10}{7}$.

Exercise 1.5. Prove that when $\frac{10}{7}$ is replaced with $\frac{8}{5}$, there are no solutions.

Exercise 1.6. Find positive integers (a, b, c, d) such that

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}} = \frac{931}{222}$$

§2 Infinite Continued Fractions

Continued fractions don't necessarily need to be finite.

Definition 2.1. An *infinite* continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} \tag{1}$$

Exercise 2.2. Prove that a continued fraction is rational if and only if it is finite.

Definition 2.3. An infinite continued fraction is *periodic* if a portion of it repeats. More formally, the continued fraction $[a_0, a_1, a_2, \dots]$ is periodic if it is of the form $[a_0, \dots, a_r, a_{r+1} m \dots a_{r+p}, a_r, a_{r+1}, \dots, a_{r+p}, \dots]$. In this case, we denote it as $[a_0, \dots, \overline{a_r, \dots, a_{r+p}}]$

Example 2.4

$[1, \overline{2}] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$ is an infinite periodic continued fraction.

Exercise 2.5. Write

$$5 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}$$

using the continued fraction notation and find its value.

Exercise 2.6. Simplify the following:

1. $[\overline{1}]$
2. $[2, \overline{4}]$
3. $[\overline{1, 2}]$

Exercise 2.7. Show that if a is positive, then

$$\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}} = 1 + \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \cdots}}}$$

Exercise 2.8. Simplify the expression

$$5 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{3 + \cdots}}}$$

Exercise 2.9. (Challenge) Prove that all periodic continued fractions are solutions to quadratics with integer coefficients.