Continued Fractions — Solutions

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§1 Introduction

Definition 1.1. A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{k-1+\frac{1}{a_k}}}}}}$$

where $a_0, a_1, \dots a_k$ are natural numbers. A continued fraction of the above form can be denoted as $[a_0, a_1, \dots a_k]$ for short.

Exercise 1.2. Simplify each of the following continued fractions:

1. $2 + \cfrac{1}{3 + \cfrac{1}{2}} = \boxed{\dfrac{16}{7}}$

2. $1 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4}}} = \boxed{\dfrac{69}{56}}$

3. $6 + \cfrac{1}{9 + \cfrac{1}{4 + \cfrac{1}{2}}} = \boxed{\dfrac{507}{83}}$

4. $9 + \cfrac{1}{12 + \cfrac{1}{21 + \cfrac{1}{2}}} = \boxed{\dfrac{4705}{518}}$

Exercise 1.3. Write each of the following as a continued fraction:

1. $0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \boxed{[0, 2, 2, 2]}$

2. $1 + \cfrac{1}{1 + \cfrac{1}{2}} = \boxed{[1, 1, 2]}$

3. $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3}}} = \boxed{[1, 2, 3, 3]}$

4. $1 + \cfrac{1}{5 + \cfrac{1}{6}} = \boxed{[1, 5, 6]}$

Exercise 1.4. $\frac{10}{7} = 1 + \cfrac{1}{2 + \cfrac{1}{3}}$, so $\boxed{[x, y, z] = [1, 2, 3]}$

Exercise 1.5. $\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$. $\boxed{\text{Clearly, the continued fraction has 4 terms not 3}}$.

Exercise 1.6. $4 + \cfrac{1}{5 + \cfrac{1}{6 + \cfrac{1}{7}}} = \frac{931}{222}$, so $\boxed{(a, b, c, d) = (4, 5, 6, 7)}$

§2 Infinite Continued Fractions

Continued fractions don't necessarily need to be finite.

Definition 2.1. An *infinite* continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} \tag{1}$$

Exercise 2.2. https://math.stackexchange.com/questions/1323586/a-real-number-is-rational-iff/-its-continued-fraction-expansion-is-finite

Definition 2.3. An infinite continued fraction is *periodic* if a portion of it repeats. More formally, the continued fraction $[a_0, a_1, a_2, \dots]$ is periodic if it is of the form $[a_0, \dots, a_r, a_{r+1}m \dots a_{r+p}, a_r, a_{r+1}, \dots, a_{r+p}, \dots]$. In this case, we denote it as $[a_0, \dots, \overline{a_r, \dots, a_{r+p}}]$

Example 2.4

$[1, \overline{2}] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$ is an infinite periodic continued fraction.

Exercise 2.5.

$$5 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}} = [5, \overline{4}]$$

$$\text{let } u = [5, \overline{4}]$$

$$u = 5 + \frac{1}{u-1} \Rightarrow (u-5)(u-1) = u^2 - 6u + 5 = 1 \Rightarrow u = \frac{6 + 2\sqrt{5}}{2} = \boxed{3 + \sqrt{5}}$$

Exercise 2.6. Simplify the following:

1. $u = 1 + \frac{1}{u} \Rightarrow u^2 + u - 1 = 0 \Rightarrow u = \boxed{\frac{\sqrt{5} - 1}{2}}$

2. $u = 2 + \frac{1}{u+2} \Rightarrow u^2 - 5 = 0 \Rightarrow u = \boxed{\sqrt{5}}$

3. $u = 1 + \cfrac{1}{2 + \cfrac{1}{u}} = 1 + \frac{u}{2u+1} \Rightarrow 2u^2 - 2u - 1 = 0 \Rightarrow u = \boxed{\frac{1 + \sqrt{3}}{2}}$

Exercise 2.7.

$$\text{let } u = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}, \text{so } u = \sqrt{a + u} \Rightarrow u^2 = a + u$$

$$\text{let } u = 1 + \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \cdots}}}, \text{so } u = 1 + \frac{a}{u} = \frac{u + a}{u} \Rightarrow u^2 = u + a$$

$$\boxed{u^2 = u + a = u^2 = a + u, \text{so they are equal}}$$

Exercise 2.8.

$$u = 5 + \cfrac{1}{3 + \cfrac{1}{u}} = 5 + \frac{u}{3u+1} \Rightarrow 3u^2 - 15u - 5 = 0 \Rightarrow u = \frac{15 \pm \sqrt{225+60}}{6} = \boxed{\frac{15+\sqrt{285}}{6}}$$

Exercise 2.9. https://sites.millersville.edu/bikenaga/number-theory/periodic-continued-fractions/periodic-continued-fractions.html