Recursion

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1 Recursion Definition

When we look at sequences of numbers, there are primarily two ways we define them. Explicitly and recursively. For example, a sequence like $a_n = n^2$ is explicit, since we can find any term by plugging in the value of n. $a_3 = 3^2 = 9$, $a_{11} = 121$, etc. On the other hand, a sequence like $F_n = F_{n-1} + F_{n-2}$, the Fibonacci Sequence, is recursive, since we need the values of the previous terms. We need to know F_7 and F_8 in order to find F_9. While it may seem a lot slower to use recursive sequences, they have widespread applications, as we will see today.

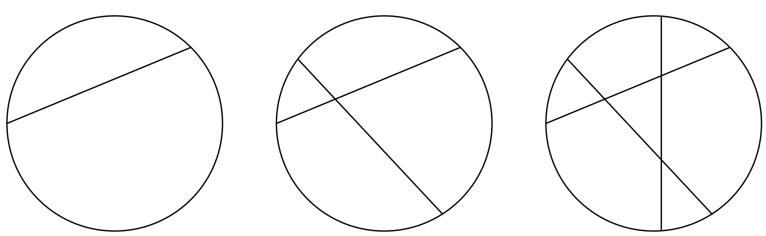
Recursion is also helpful for certain types of challenging problems, especially in competitions. You are usually given a counting problem that has an unreasonably large number. For example: find the sum of the integers $1 + 2 + 3 + \cdots + 100$. If that number were smaller, like if you only had to add from 1 to 10, then this problem would be easy. But when you have to find the answer to this problem with a very large number, you will instead have to find a shortcut that will save you from doing all the brute force work. One way you could do this is to first try solving this same problem for small numbers. What is $1 + 2$ or $1 + 2 + 3$ or $1+2+3+4$? Once you have enough small cases, maybe you will be able to find a pattern. In this case, you might just discover that the sum of integers $1 + 2 + \cdots + n$ is equal to $\frac{n(n+1)}{2}$. So if you want to find the sum of integers from 1 to 100, you could do it easily. This is the idea we will explore with these following recursion problems. These problems are hard because one of the numbers is very large; we will try solving the problem with small numbers and look for a pattern that will allow us to answer the original.

2 Recursion Practice

Remember, the key is this: **start small.** If this problem is intimidating, and a lot of them are, ask yourself: if I could make a number in the problem smaller, would this problem be easier? Start small, and try and find a pattern.

1. You want to create an answer key for a 10-question true-or-false test, but you want to avoid having two falses in a row. How many ways can you create this answer key?

2. If I make one straight cut in a circular pizza, I get 2 pieces. With 2 cuts, I can have at most 4 pieces. With 3 cuts, I can have at most 7. Examples of 1, 2, and 3 cuts are shown below.



   Find the maximum number of pieces I can cut from a circular pizza with 10 straight cuts.

3. Refer to problem 2. What is the maximum number of pieces that a pizza can be cut into with 100 straight cuts?

4. Evaluate this sum:
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{99 \cdot 100}.$$

5. Everyday at school, Jo climbs a flight of 8 stairs. Jo can take the stairs 1, 2, or 3 at a time. Forexample, Jo could climb 3, then 1, then 2. In how many ways can Jo climb the stairs?

6. How many ways can I create an answer key to a true/false quiz with 10 questions such that there are never three "false" answers next to each other?

7. A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and
$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$
for all $n \geq 3$. What is a_{2022}?

8. Call a set of integers "spacy" if there are no pairs of numbers in that set that are less than 3 apart. How many subsets of $\{1, 2, 3, \ldots, 12\}$, *including the empty set*, are spacy? (Try finding the number of spacy subsets of $\{1, \ldots, n\}$ for small values of n. I suggest taking values until $n = 6$ to see the pattern)

   (A) 121 (B) 123 (C) 125 (D) 127 (E) 129

3 Fibonacci

Let's take a look at one of the most famous recursive sequences of numbers in mathematics: the Fibonacci Sequence.

1. The first few terns in the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13. What is the pattern? Can you write the rule for the recursion?

2. Find the next 9 terms in the sequence, and write them below.

4 The Golden Ratio and Binet's Formula

You should now have the first 16 terms of the Fibonacci Sequence written. Now, try dividing the 2nd term by the 1st, the 3rd by the 2nd, the 4th by the 3rd, and so on. Can find a pattern of any kind?

The value you obtain from these divisions is called the golden ratio, usually written as ϕ, and it is approximately 1.618. It is a special value that can be seen in many different aspects of life, including the Statue of Liberty. Since all these divisions have the same ratio, they can form a closed form power function. The closed form, known as Binet's Formula, is complicated, but here it is for those who are curious.

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

We see the golden ratio in this formula as well. Specifically, $\frac{1+\sqrt{5}}{2}$ is the exactly value of ϕ, the golden ratio. Based on this formula, see if you can show why dividing terms of the Fibonacci Sequence gives you the golden ratio. (Hint: $\frac{1-\sqrt{5}}{2}$ is small, and gets smaller as n increases.)

5 Fibonacci in Pascal's Triangle

Draw Pascal's triangle and identify the Fibonacci numbers that were noticed earlier. Extra instructions for this will be given in class. Try and look for any patterns.