Cyclic Quadrilaterals

Pleasanton Math Circle

1 Theory and Examples

Theorem 1.1 (Inscribed Angle Theorem). If A, B, C lie on a circle, then $\angle ACB$ subtends an arc of measure $2\angle ACB$.

Proposition 1.2 (Cyclic Quadrilaterals). Let $ABCD$ be a convex quadrilateral. Each of the three statements below are equivalent.

1. $ABCD$ is cyclic.
2. $\angle ACB = \angle ADB$.
3. $\angle ABC + \angle CDA = 180$.

(a) $\angle ACB = \angle ADB$ (b) $\angle ACB = \angle ADB$

Figure 1: Property of Cyclic Quadrilaterals

Now we can prove the existence of the first Fermat point.

Theorem 1.3 (Fermat Point). Given $\triangle ABC$, construct equilateral triangles $\triangle BCD, \triangle CAE, \triangle ABF$ outside of $\triangle ABC$. Then AD, BE, CF concur at the **first Fermat point**.

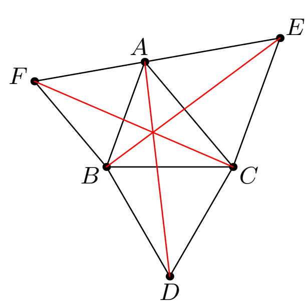

Figure 2: The Fermat Point

Proof. First we show that the circles $\odot(BCD), \odot(CAE), \odot(ABF)$ share a common point. Let $\odot(ABF), \odot(CAE)$ meet at F_1. Then $\angle AF_1B = \angle CF_1A = 120°$. Therefore, $\angle BF_1C = 120 = 180 - \angle BDC$, so BF_1CD is cyclic as desired. Now notice that $\angle AF_1C = 120 = 180 - 60 = 180 - \angle DBC = 180 - \angle DF_1C$. So A, F_1, D are collinear and the proof follows. □

2 Exercises

Exercise 2.1. Consider a circle with diameter AB. Then C is on this circle if and only if $\angle ACB = 90°$.

Exercise 2.2. In $\triangle ABC$, let AD, BE, CF be altitudes meeting at the orthocenter H. Find 6 quadruples of points in this configuration that are concyclic.

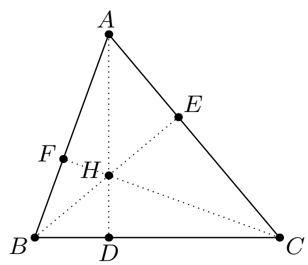

Figure 3: Orthocenter

Exercise 2.3. In Figure 3, show that $\angle HBC = 90 - \angle C$ and $\angle HCB = 90 - \angle B$. Deduce that $\angle BHC = 180 - \angle A$.

Exercise 2.4. Use Exercise 2.3 to show that the reflection of H across BC lies on the circumcircle of $\triangle ABC$.

Theorem 2.5 (Miquel's Theorem)**.** In $\triangle ABC$, choose points D, E, F on sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Then circles $\odot(AEF), \odot(BFD), \odot(CDE)$ share a common point.

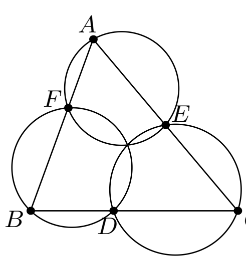

Figure 4: Miquel's Theorem

Exercise 2.6. Let $\odot(AEF)$ and $\odot(BFD)$ meet at a point M. Show that $\angle EMF = 180 - \angle A$ and $\angle FMD = 180 - \angle B$. Using this, find $\angle DME$.

Exercise 2.7. Using Exercise 2.6, show that M lies on $\odot(CDE)$. Deduce Miquel's Theorem.

Theorem 2.8 (Reim's Theorem)**.** Choose points A, B, X, Y on circle ω_1 and let C and D be points on AX and BY. Then $AB \parallel CD$ if X, Y, C, D are concyclic.

Exercise 2.9. In Figure 5 show that $\angle ABY = 180 - \angle CDY$ to deduce Reim's Theorem.

Theorem 2.10 (Simson Line)**.** Let P be a point on $\odot(ABC)$. Let D, E, F be the feet of the perpendiculars from P to $\overline{BC}, \overline{CA}, \overline{AB}$. Prove that D, E, F are collinear. This line is known as the **Simson Line**. Hint: Prove that $\angle PEF = 180 - \angle PED$.

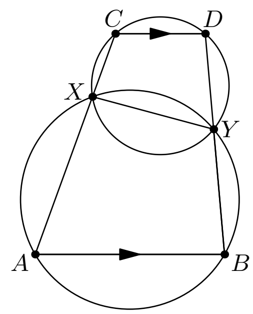

Figure 5: Reim's Theorem

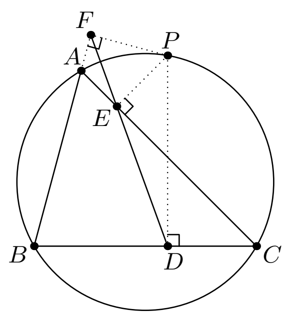

Figure 6: Simson Line

Exercise 2.11. In Figure 6 show that $AEPF$ and $CDEP$ are cyclic.

Exercise 2.12. Now prove that $\angle PEF = 180 - \angle PED$ and deduce the existence of the Simson Line.

Theorem 2.13 (Ptolemy's Theorem)**.** Given cyclic quadrilateral $ABCD$, the product of the diagonals is equal to the sum of the products of the opposite sides. Equivalently,

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

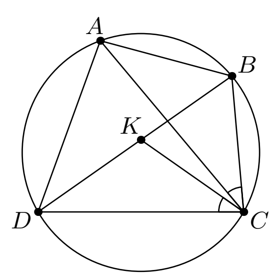

Figure 7: Ptolemy's Theorem

Let K be on BD such that $\angle KCD = \angle ACB$.

Exercise 2.14. Show that $\triangle DKC \sim \triangle ABC$ and $\triangle KBC \sim \triangle DAC$.

Exercise 2.15. Now show that $\frac{KD}{CD} = \frac{AB}{AC}$ and $\frac{KB}{BC} = \frac{DA}{AC}$. Deduce Ptolemy's Theorem.