Cyclic Quadrilaterals

Pleasanton Math Circle

1 Theory and Examples

Theorem 1.1 (Inscribed Angle Theorem). If $A, B, C$ lie on a circle, then $\angle ACB$ subtends an arc of measure $2\angle ACB$.

Proposition 1.2 (Cyclic Quadrilaterals). Let $ABCD$ be a convex quadrilateral. Each of the three statements below are equivalent.

1. $ABCD$ is cyclic.
2. $\angle ACB = \angle ADB$.
3. $\angle ABC + \angle CDA = 180$.

![Figure 1: Property of Cyclic Quadrilaterals](image)

Now we can prove the existence of the first Fermat point.

Theorem 1.3 (Fermat Point). Given $\triangle ABC$, construct equilateral triangles $\triangle BCD, \triangle CAE, \triangle ABF$ outside of $\triangle ABC$. Then $AD, BE, CF$ concur at the first Fermat point.

![Figure 2: The Fermat Point](image)
Proof. First we show that the circles \( \odot(BCD), \odot(CAE), \odot(ABF) \) share a common point. Let \( \odot(ABF), \odot(CAE) \) meet at \( F_1 \). Then \( \angle AF_1B = \angle CF_1A = 120^\circ \). Therefore, \( \angle BF_1C = 120 = 180 - \angle BDC \), so \( BF_1CD \) is cyclic as desired. Now notice that \( \angle AF_1C = 120 = 180 - 60 = 180 - \angle DBC = 180 - \angle DF_1C \). So \( A, F_1, D \) are collinear and the proof follows.

2 Exercises

Exercise 2.1. Consider a circle with diameter \( AB \). Then \( C \) is on this circle if and only if \( \angle ACB = 90^\circ \).

Exercise 2.2. In \( \triangle ABC \), let \( AD, BE, CF \) be altitudes meeting at the orthocenter \( H \). Find 6 quadruples of points in this configuration that are concyclic.

**Figure 3: Orthocenter**

Exercise 2.3. In Figure 3 show that \( \angle HBC = 90 - \angle C \) and \( \angle HCB = 90 - \angle B \). Deduce that \( \angle BHC = 180 - \angle A \).

Exercise 2.4. Use Exercise 2.3 to show that the reflection of \( H \) across \( BC \) lies on the circumcircle of \( \triangle ABC \).

Theorem 2.5 (Miquel’s Theorem). In \( \triangle ABC \), choose points \( D, E, F \) on sides \( BC, CA, AB \) respectively. Then circles \( \odot(AEF), \odot(BFD), \odot(CDE) \) share a common point.

**Figure 4: Miquel’s Theorem**

Exercise 2.6. Let \( \odot(AEF) \) and \( \odot(BFD) \) meet at a point \( M \). Show that \( \angle EMF = 180 - \angle A \) and \( \angle FMD = 180 - \angle B \). Using this, find \( \angle DME \).

Exercise 2.7. Using Exercise 2.6 show that \( M \) lies on \( \odot(CDE) \). Deduce Miquel’s Theorem.

Theorem 2.8 (Reim’s Theorem). Choose points \( A, B, X, Y \) on circle \( \omega_1 \) and let \( C \) and \( D \) be points on \( AX \) and \( BY \). Then \( AB \parallel CD \) if \( X, Y, C, D \) are concyclic.

Exercise 2.9. In Figure 5 show that \( \angle ABY = 180 - \angle CDY \) to deduce Reim’s Theorem.

Theorem 2.10 (Simson Line). Let \( P \) be a point on \( \odot(ABC) \). Let \( D, E, F \) be the feet of the perpendiculars from \( P \) to \( BC, CA, AB \). Prove that \( D, E, F \) are collinear. This line is known as the Simson Line. Hint: Prove that \( \angle PEF = 180 - \angle PED \).
Exercise 2.11. In Figure 6 show that $AEPF$ and $CDEP$ are cyclic.

Exercise 2.12. Now prove that $\angle PEF = 180 - \angle PED$ and deduce the existence of the Simson Line.

Theorem 2.13 (Ptolemy’s Theorem). Given cyclic quadrilateral $ABCD$, the product of the diagonals is equal to the sum of the products of the opposite sides. Equivalently,

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Let $K$ be on $BD$ such that $\angle KCD = \angle ACB$.

Exercise 2.14. Show that $\triangle DKC \sim \triangle ABC$ and $\triangle KBC \sim \triangle DAC$.

Exercise 2.15. Now show that $\frac{KD}{CD} = \frac{AB}{AC}$ and $\frac{KB}{BC} = \frac{DA}{AC}$. Deduce Ptolemy’s Theorem.