1 Introduction

Welcome to the PMC meeting this week! In today's meeting, we will be learning all about prime numbers and applying them to some interesting problems.

2 Warm-Up

A prime number is a whole number greater than 1 whose only factors are 1 and itself. A "factor" is considered a whole number that can be divided evenly into another whole number. Let's look at some examples:

- **7**: 7's only factors are 1 and 7. No other whole number divides 7 evenly.
- **5**: The only factors of 5 are 5 and 1. (Note that 2.5 is not considered a factor because it is not a whole number).
- **13**: The only factors of 13 are 13 and 1.

Try this: List the first ten prime numbers.

In contrast, a composite number is a positive integer that has at least one divisor other than 1 and itself. Essentially, it is the opposite of a prime number. (**Note**: 0 and 1 are neither prime nor composite numbers!) Some examples of composite numbers include:

- **4**: The factors of 4 are 4, 1, and 2.
- **21**: The factors of 21 are 21, 1, 7, and 3.
- **15**: The factors of 15, 1, 3, and 5.

Try this: List the first ten composite numbers.

3 Sieve of Eratosthenes

One method for finding prime numbers is by using the **Sieve of Eratosthenes**. Here are the steps to this algorithm, using the table below:

1. Cross out 1 (it is not prime).
2. Circle 2 (it is prime) and then cross out all multiples of 2.
3. Circle 3 (it is prime) and then cross out all multiples of 3.
4. Circle 5, then cross out all multiples of 5.
5. Circle 7, then cross out all multiples of 7.
6. Continue by circling the next number not crossed out, then cross out all of its multiples.

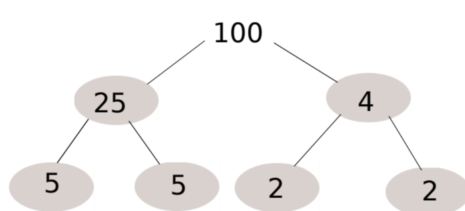
The circled numbers are all the prime numbers less than 100. This method is called the Sieve of Eratosthenes because of the way in which we begin with many numbers and sift many away, leaving only the primes.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

4 Prime Factorization

Any integer greater than 1 can be expressed as the product of prime numbers. For example, $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$. To find a number's prime factorization, find any of its factors. Then, break the factors down until they are prime numbers.

Example: For 100, we know that $100 = 25 \cdot 4$. We can create a factor tree to perform the prime factorization of 100, as shown below:



Try it! Find the prime factorization of each of the following numbers:

1. 21
2. 64
3. 75
4. 97
5. 478
6. 816

5 GCD

The **greatest common divisor (GCD)** of two numbers is the greatest number that is a factor of both numbers. For example, the GCD of 24 and 15 is 3. ($3 \cdot 8 = 24$ and $3 \cdot 5 = 15$)
How to find the GCD of two numbers:

1. Find the prime factorizations of both numbers.
2. Multiply the common prime factors together.
3. The product is the GCD.

For example, let's find the GCD of 24 and 36:
$24 = 2 \cdot 2 \cdot 2 \cdot 3$
$36 = 2 \cdot 2 \cdot 3 \cdot 3$
The prime factorizations of 24 and 36 have the following factors in common: 3,2,2. So, the GCD of 24 and 36 is $2 \cdot 2 \cdot 3 = 12$.
Practice:

1. Find the GCD of 36 and 75.
2. Find the GCD of 105 and 84.

6 LCM

We know that a multiple of a number is that number multiplied by any whole number. A common multiple of two numbers is a number that is a multiple of both numbers. The **least common multiple (LCM)** is the smallest number that is a multiple of both numbers.
How to find the LCM of two numbers:

1. Find the prime factorizations of both numbers.
2. Multiply the prime factors the greatest number of times they appear in either number.
3. The product is the LCM.

For example, let's find the LCM of 9 and 12:
$9 = 3 \cdot 3$
$12 = 2 \cdot 2 \cdot 3$
3 appears twice in 9, which is greater than the one time it appears in 12. 2 appears twice in 12. So, the LCM of 9 and 12 is $3 \cdot 3 \cdot 2 \cdot 2 = 36$.
Practice:

1. Find the LCM of 7 and 15.
2. Find the LCM of 8 and 14.

7 How Many Primes are There?

First, you can give a guess. Are there an infinite number of primes? In other words, is there a largest prime number? Circle **Y** or **N**

To start, we know that no whole number greater than 1 is a factor of 1. If you have two numbers a and b greater than 1 such that a divides b, it would then follow that a does not divide b+1.

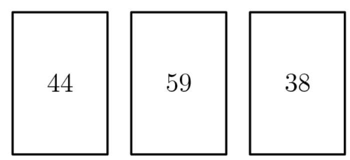
Can you prove that there are an infinite number of primes? or not? This is a tough idea, so feel free to ask for help.

8 Interesting Problems

1. (2014 AMC 8 #4) The sum of two prime numbers is 85. What is the product of these two prime numbers?
   (A) 85 **(B)** 91 **(C)** 115 **(D)** 133 **(E)** 166

2. The prime numbers between 10 and 20 are added together to form the number Q. What is the largest prime divisor of Q?

3. (2006 AMC 8 #25) Barry wrote 6 different numbers, one on each side of 3 cards, and laid the cards on a table, as shown. The sums of the two numbers on each of the three cards are equal. The three numbers on the hidden sides are prime numbers. What is the average of the hidden prime numbers?



   (A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

4. (2016 AMC 8 #20) The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?
   (A) 20 **(B)** 30 **(C)** 60 **(D)** 120 **(E)** 180

5. Which of the following expressions is never a prime number when p is a prime number?
   (A) $p^2 + 16$ **(B)** $p^2 + 24$ **(C)** $p^2 + 26$ **(D)** $p^2 + 46$ **(E)** $p^2 + 96$

6. A perfect square is the number that you get when you square another number. Some examples of perfect squares are 1, 4, 9, 16 and 25. What is the LCM of 14 and 18 that is also a perfect square?